Beyond Bird's Nested Arrays I

My various Array Notations use curly brackets { } for the entire array, square brackets [ ] for 'separator' arrays (markers denoting the end of the linear, planar or higher dimensional space within the array) and angle brackets ‹ › to denote that a space of a certain dimension is filled up with entries of a certain integer. In order to proceed beyond my Nested Array Notation, I have managed to link the separator arrays with the various ordinal infinities or transfinite numbers, in order to make it easier for mathematicians to follow (quite a large list of them).

The first epsilon ordinal,

$\varepsilon_0 = \omega\text{^^}\omega$
$\quad = \omega\text{^}\omega\text{^}\omega\text{^}...\text{^}\omega$ (with ω omegas).

The next epsilon numbers are ε_1, ε_2, ε_3 etc. Using ε_α to denote the (α+1)th epsilon number (as α is a Greek letter variable, it may be either finite or transfinite), $\varepsilon_{\alpha+1}$ is the limit of the sequence,

$\varepsilon_\alpha+1$,
$\omega\text{^}(\varepsilon_\alpha+1)$,
$\omega\text{^}\omega\text{^}(\varepsilon_\alpha+1)$,

etc., so,

$\varepsilon_{\alpha+1} = \omega\text{^}\omega\text{^}\omega\text{^}...\text{^}\omega\text{^}(\varepsilon_\alpha+1)$ (with ω omegas).

The sequence begins with $\varepsilon_\alpha+1$ rather than ε_α since $\omega\text{^}\varepsilon_\alpha = \varepsilon_\alpha$. Putting it another way,

$\varepsilon_{\alpha+1} = \varepsilon_\alpha\text{^^}\omega$.

Since,

$\omega\text{^^}(\omega+1) = \omega\text{^}(\omega\text{^^}\omega) = \omega\text{^}\varepsilon_0 = \varepsilon_0$,

I have redefined Knuth's Up-arrow Notation with transfinite values so that

$\omega\text{^^}(\omega+1) = \omega\text{^}(\varepsilon_0+1)$,

and similarly, $\omega\text{^^}\beta = \varepsilon_\alpha$ implies that $\omega\text{^^}(\beta+1) = \omega\text{^}(\varepsilon_\alpha+1)$, in order to achieve a strictly increasing function for f(β) = ω^^^···^β (with γ up-arrows, for some finite or transfinite γ). Using this redefinition,

$\varepsilon_1 = \omega\text{^^}(\omega2)$,
$\varepsilon_2 = \omega\text{^^}(\omega3)$,

and, in general,

$\varepsilon_\alpha = \omega\text{^^}(\omega(1+\alpha))$ (1+α = α for all α ≥ ω).

I find that,

$\varepsilon(\varepsilon_0) = \omega\text{^^}\omega\text{^^}\omega$,
$\varepsilon(\varepsilon(\varepsilon_0)) = \omega\text{^^}\omega\text{^^}\omega\text{^^}\omega$,

etc., so, the first ordinal beyond the epsilon numbers, zeta-0 is

$\zeta_0 = \varepsilon(\varepsilon(\varepsilon(...(\varepsilon_0)...)))$ (with ω epsilons)
$\quad = \omega\text{^^^}\omega$.

$\zeta_{\alpha+1}$ is the limit of the sequence,

$\zeta_\alpha+1$,
$\varepsilon(\zeta_\alpha+1)$,
$\varepsilon(\varepsilon(\zeta_\alpha+1))$,

etc., so,

$\zeta_{\alpha+1} = \varepsilon(\varepsilon(\varepsilon(...(\varepsilon(\zeta_\alpha+1))...)))$ (with ω epsilons)
$\quad = \omega\text{^^}\omega\text{^^}...\text{^^}\omega\text{^^}(\zeta_\alpha+1)$ (with ω omegas),

when $\omega\text{^^^}\beta = \zeta_\alpha$ implies that $\omega\text{^^^}(\beta+1) = \varepsilon(\zeta_\alpha+1)$ rather than $\omega\text{^^^}(\beta+1) = \varepsilon(\zeta_\alpha) = \zeta_\alpha$.

Hence,

$\zeta_1 = \omega\text{^^^}(\omega 2)$,

$\zeta_2 = \omega\text{^^^}(\omega 3)$,

and, in general,

$\zeta_\alpha = \omega\text{^^^}(\omega(1+\alpha))$.

$\zeta(\zeta_0) = \omega\text{^^^}\omega\text{^^^}\omega$,

$\zeta(\zeta(\zeta_0)) = \omega\text{^^^}\omega\text{^^^}\omega\text{^^^}\omega$,

etc., so, the first ordinal beyond the zeta numbers, eta-0 is

$\eta_0 = \zeta(\zeta(\zeta(\ldots(\zeta_0)\ldots)))$ (with ω zetas)

$= \omega\text{^^^^}\omega$.

There are more expressive ordinal notations, for example,

$\varphi(0) = 1$,

$\varphi(1) = \omega$,

$\varphi(\alpha) = \omega\text{^}\alpha$,

$\varphi(1, \alpha) = \varepsilon_\alpha$,

$\varphi(2, \alpha) = \zeta_\alpha$,

$\varphi(3, \alpha) = \eta_\alpha$,

where φ (lower case Greek letter phi) is known as the Veblen function. It is defined as follows:

$\varphi(\alpha+1, 0) = \varphi(\alpha, \varphi(\alpha, \varphi(\ldots \varphi(\alpha, 0) \ldots)))$ (with ω φ's),

$\varphi(\alpha+1, \beta+1) = \varphi(\alpha, \varphi(\alpha, \varphi(\ldots \varphi(\alpha+1, \beta)+1 \ldots)))$ (with ω φ's).

φ(1, α) represents the epsilon numbers, φ(2, α) the zeta numbers, and so on. φ(α, β) (also written as $\varphi_\alpha(\beta)$) represents the (α-1)th set of numbers (after the epsilons).

In Knuth's Up-arrow Notation and my Linear Array Notation, for α ≥ 1,

$\varphi(\alpha, 0) = \omega\text{^^^}^{\cdots}\text{^}\omega$ (with 1+α up-arrows)

$= \{\omega, \omega, 1+\alpha\}$,

$\varphi(\alpha, \beta) = \omega\text{^^^}^{\cdots}\text{^}(\omega(1+\beta))$ (with 1+α up-arrows)

$= \{\omega, \omega(1+\beta), 1+\alpha\}$.

The limit of the sequence,

0,

$\varphi(0) = 1$,

$\varphi(\varphi(0), 0) = \varphi(1, 0) = \varepsilon_0 = \{\omega, \omega, 2\}$,

$\varphi(\varphi(\varphi(0), 0), 0) = \varphi(\varepsilon_0, 0) = \{\omega, \omega, \varepsilon_0\} > \{\omega, \omega, \omega\} = \{\omega, 2, 1, 2\}$,

$\varphi(\varphi(\varphi(\varphi(0), 0), 0), 0) > \{\omega, \omega, \{\omega, \omega, \omega\}\} = \{\omega, 3, 1, 2\}$,

$\varphi(\varphi(\varphi(\varphi(\varphi(0), 0), 0), 0), 0) > \{\omega, 4, 1, 2\}$,

etc., is

$\varphi(1, 0, 0) = \varphi(\varphi(\varphi(\ldots \varphi(0) \ldots), 0), 0)$ (with ω φ's)

$= \{\omega, \omega, 1, 2\}$.

$= \omega \{\omega \{\omega \{ \ldots \{\omega \{\omega\} \omega\} \ldots \} \omega\} \omega\} \omega$ (with ω layers of curly brackets).

φ(1, 0, 0) is the Feferman-Schütte ordinal and normally written as Γ_0 (Γ is the capital Greek letter Gamma). We really needed a special notation well before even thinking of this ordinal, for we would truly run out of letters if we used epsilon, zeta, eta etc. While ε_0 represents the first step of reaching this ordinal, the second step would require ε_0 letters, the third step would require as many letters as the ordinal represented by the second step, and so on. And ω of these steps are needed to get to Γ_0.

We can generalise for any number of arguments of φ.

$$\varphi(R,\alpha+1,0) = \varphi(R,\alpha,\ \varphi(R,\alpha,\ \varphi(\ ....\ \varphi(R,\alpha,0)\ ....\ ))),$$
$$\varphi(R,\alpha+1,\beta+1) = \varphi(R,\alpha,\ \varphi(R,\alpha,\ \varphi(\ ....\ \varphi(R,\alpha+1,\beta)+1\ ....\ ))),$$
$$\varphi(R,\alpha+1,0,0,..,0) = \varphi(R,\alpha,\ \varphi(R,\alpha,\ \varphi(\ ....\ \varphi(R,\alpha,0,0,..,0)\ ....\ ),\ 0,..,0),\ 0,..,0),$$
$$\varphi(R,\alpha+1,0,..,0,\beta+1) = \varphi(R,\alpha,\ \varphi(R,\alpha,\ \varphi(\ ....\ \varphi(R,\alpha+1,0,..,0,\beta)+1\ ....\ ),\ 0,..,0),\ 0,..,0),$$

where R represents a string of the remaining arguments of φ and 0,..,0 is a string of one or more zeroes. There are ω φ's on the right hand side of each of the four equations.
This was adapted from page 29 of "Why Ordinals are Good for You" (http://cl-informatik.uibk.ac.at/~georg/publications/esslli03.pdf).

When λ is a limit ordinal (of the form ωα for α > 0) and α < λ,

$\varphi(R,\ \alpha,\ \varphi(R,\ \lambda,\ \beta)+1)$ has limit ordinal $\varphi(R,\ \lambda,\ \beta+1)$ as α tends to λ,

$\varphi(R,\alpha,0,..,0,\varphi(R,\lambda,0,..,0,\beta)+1)$ has limit ordinal $\varphi(R,\lambda,0,..,0,\beta+1)$ as α tends to λ,

where R represents a string of the remaining arguments of φ and 0,..,0 is a string of one or more zeroes.

I found that,

$$\begin{aligned}\varphi(\alpha,\ 0,\ 0,\ \dots,\ 0) &= \{\omega,\ \omega,\ 1,\ \dots,\ 1,\ 1+\alpha\}\\ &= \{\omega,\ \omega,\ \omega,\ \dots,\ \{\omega,\ \omega\text{-}1,\ 1,\ \dots,\ 1,\ 1+\alpha\},\ \alpha\}\\ &> \{\omega,\ \omega,\ \omega,\ \dots,\ \omega,\ \alpha\},\end{aligned}$$

with the dots representing the same number of arguments, and, for ordinals above $\varphi(1,\ 0) = \varepsilon_0$,

$$\varphi(\alpha_n,\ \dots,\ \alpha_3,\ \alpha_2,\ \alpha_1) = \{\omega,\ \omega(1+\alpha_1),\ 1+\alpha_2,\ 1+\alpha_3,\ \dots,\ 1+\alpha_n\}.$$

The limit of the sequence,

$\varphi(1) = \omega$,
$\varphi(1,\ 0) = \varepsilon_0$,
$\varphi(1,\ 0,\ 0) = \Gamma_0$,
$\varphi(1,\ 0,\ 0,\ 0)$,
$\varphi(1,\ 0,\ 0,\ 0,\ 0)$,

etc., is

$\theta(\Omega\text{^}\omega) = \varphi(1,\ 0,\ 0,\ \dots,\ 0)$ (with ω zeroes),

which is the small Veblen ordinal. The θ (theta) function is an ordinal collapsing function and Ω (capital Greek letter Omega) denotes the first uncountable ordinal.

$\theta(\Omega\text{^}\omega) = \{\omega,\ \omega\ [2]\ 2\} = \{\omega,\ \omega,\ \omega,\ \dots,\ \omega\}$ (with ω omegas).

Relationship between the θ and φ functions:

$\theta(0) = \varphi(0) = 1$,	
$\theta(1) = \varphi(1,\ 0) = \varepsilon_0$,	
$\theta(2) = \varphi(2,\ 0) = \zeta_0$,	
$\theta(\alpha) = \varphi(\alpha,\ 0)$	(α < Ω),
$\theta(\omega) = \varphi(\omega,\ 0)$,	
$\theta(\Omega) = \theta(\theta(\dots\theta(0)\dots))$	(with ω θ's)
$= \varphi(1,\ 0,\ 0) = \Gamma_0$,	
$\theta(\Omega\alpha) = \varphi(\alpha,\ 0,\ 0)$	(α < Ω),
$\theta(\Omega\text{^}2) = \varphi(1,\ 0,\ 0,\ 0)$	(Ackermann ordinal),
$\theta(\Omega\text{^}n) = \varphi(1,\ 0,\ 0,\ \dots,\ 0)$	(with n+1 zeroes),
$\theta(\Omega\text{^}\omega) = \varphi(1,\ 0,\ 0,\ \dots,\ 0)$	(with ω zeroes).

The notation for the ordinal collapsing function may vary. Texts often use ψ (psi) ordinals as follows:

$\psi(0) = \varphi(1, 0) = \varepsilon_0$,
$\psi(1) = \varphi(1, 1) = \varepsilon_1$,
$\psi(\alpha) = \varphi(1, \alpha) = \varepsilon_\alpha$ $(\alpha < \Omega)$,
$\psi(\Omega) = \varphi(2, 0) = \zeta_0$,
$\psi(\Omega 2) = \varphi(2, 1) = \zeta_1$,
$\psi(\Omega\alpha) = \varphi(2, \alpha\text{-}1) = \zeta_{\alpha-1}$ $(1 \le \alpha < \omega)$
$\quad = \varphi(2, \alpha) = \zeta_\alpha$ $(\omega \le \alpha < \Omega)$,
$\psi(\Omega\text{^}2) = \varphi(3, 0)$,
$\psi(\Omega\text{^}\alpha) = \varphi(1+\alpha, 0)$ $(1 \le \alpha < \Omega)$,
$\psi((\Omega\text{^}\alpha)\beta) = \varphi(1+\alpha, \beta\text{-}1)$ $(1 \le \alpha < \Omega, 1 \le \beta < \omega)$
$\quad = \varphi(1+\alpha, \beta)$ $(1 \le \alpha < \Omega, \omega \le \beta < \Omega)$,
$\psi(\Omega\text{^}\Omega) = \varphi(1, 0, 0) = \Gamma_0$,
$\psi(\Omega\text{^}(\Omega\alpha)) = \varphi(\alpha, 0, 0)$ $(1 \le \alpha < \Omega)$,
$\psi(\Omega\text{^}\Omega\text{^}2) = \varphi(1, 0, 0, 0)$,
$\psi(\Omega\text{^}\Omega\text{^}n) = \varphi(1, 0, 0, \ldots, 0)$ (with n+1 zeroes, $n \ge 1$),
$\psi(\Omega\text{^}\Omega\text{^}\omega) = \varphi(1, 0, 0, \ldots, 0)$ (with ω zeroes).

While $\theta(\Omega\text{^}\omega) = \psi(\Omega\text{^}\Omega\text{^}\omega)$ is known as the small Veblen ordinal, $\theta(\Omega\text{^}\Omega) = \psi(\Omega\text{^}\Omega\text{^}\Omega)$ is known as the large Veblen ordinal, which is

$\theta(\Omega\text{^}\theta(\Omega\text{^}\ldots\theta(\Omega)\ldots))$ (with ω θ's).

Larger ordinals exist, for example,

$\theta(\Omega\text{^}\Omega\text{^}\omega)$,
$\theta(\Omega\text{^}\Omega\text{^}\Omega)$

and even

$\theta(\varepsilon_{\Omega+1}) = \theta(\Omega\text{^}\Omega\text{^}\Omega\text{^}\ldots)$ (with ω Ω's),

which is known as the Bachmann-Howard ordinal.

My Nested Array Notation can be extended beyond the ε_0 recursion level by introducing 'superseparators', starting with [[1]], which separates ε_0-spaces and has level ε_0. I can define

$\{a, b\ [[1]]\ 2\} = \{a \langle\langle 0 \rangle\rangle\ b\}$
$\quad = \{a \langle b \langle b \langle \ldots \langle b \langle b \rangle\ b \rangle \ldots \rangle\ b \rangle\ b \rangle\ b\}$ (with b b's from centre to right),

which is equivalent to

$\{a, b\ [1\ [1\ [\ \ldots\ [1\ [1, 2]\ 2]\ \ldots\ ]\ 2]\ 2]\ 2\}$ (with b-1 square brackets, for $b \ge 2$).

This function is ε_0-recursive, since ε_0 is the limit of the sequence $\omega\text{^}\omega$, $\omega\text{^}\omega\text{^}\omega\text{^}\omega$, $\omega\text{^}\omega\text{^}\omega\text{^}\omega\text{^}\omega\text{^}\omega$ etc., or $\omega\text{^^}2$, $\omega\text{^^}4$, $\omega\text{^^}6$ etc. We need $\varepsilon_0\text{^^}\omega$ to achieve ε_1.

After [[1]] comes [[2]], [[3]], ... , [[1, 2]], ... , [[1 [2] 2]], ... [[1 [1, 2] 2]], and so on. [[n+1]] separates (ε_0+n)-spaces, [[m+1, n+1]] separates (ε_0+ωn+m)-spaces etc. If [X] is a separator with level α (a combination of a finite number and the various ordinal infinities, as X is a string of characters) then [[X]] is a separator with level $\varepsilon_0+\alpha$, meaning that it can handle up to $\omega\text{^}(\varepsilon_0+\alpha) = \varepsilon_0(\omega\text{^}\alpha)$ arguments. When X 'fills-up' an ε_0-space, I need to nest the [[1]] separator inside double square brackets. The $\varepsilon_0 2$ level [[1 [[1]] 2]] separator is defined as follows:

$\{a, b\ [[1\ [[1]]\ 2]]\ 2\} = \{a \langle\langle 0\ [[1]]\ 2 \rangle\rangle\ b\}$
$\quad = \{a \langle\langle b \langle\langle 0 \rangle\rangle\ b \rangle\rangle\ b\}$
$\quad = \{a \langle\langle b \langle b \langle \ldots \langle b \langle b \rangle\ b \rangle \ldots \rangle\ b \rangle\ b \rangle\rangle\ b\}$
(with b b's from centre to right within the double angle brackets).

In general,

{a, b [[1 [[1]] n #]] 2} = {a ‹‹0 [[1]] n #›› b}
= {a ‹‹ b ‹‹0›› b [[1]] n-1 # ›› b}
= {a ‹‹ b ‹b ‹ ... ‹b ‹b› b› ... › b› b [[1]] n-1 # ›› b}
(with b b’s from centre to right within the double angle brackets),

where # is a ‘wildcard’ character representing the remainder of the array (which can be an empty string).

[[1 [[1]] 2]] has level $\varepsilon_0$2,
[[X [[1]] 2]] has level $\varepsilon_0$2+α,
[[1 [[1]] 3]] has level $\varepsilon_0$3,
[[1 [[1]] 1, 2]] has level ε_0ω,
[[1 [[1]] 1 [2] 2]] has level ε_0(ω^ω),
[[1 [[1]] X]] has level ε_0α.

At this point, I use the square outline symbol (□) in place of the [[1]] separator, in order to make some of the expressions easier to read. With k 1’s and □ symbols inside the double square brackets,

{a, b [[1 □ 1 □ ... □ 1 □ n #]] 2} = {a ‹‹ 0 □ 1 □ ... □ 1 □ n # ›› b}
= {a ‹‹ S □ S □ ... □ S □ n-1 # ›› b} (with k S strings),

where S = ‘b ‹‹0›› b’
= ‘b ‹b ‹ ... ‹b ‹b› b› ... › b› b’ (with b b’s from centre out)

and # is rest of array.

[[1 □ 1 □ 2]] has level ε_0^2,
[[1 □ 1 □ 3]] has level (ε_0^2)2,
[[1 □ 1 □ X]] has level (ε_0^2)α,
[[1 □ 1 □ 1 □ 2]] has level ε_0^3,
[[1 □ 1 □ ... □ 1 □ 2]] (with n 1’s) has level ε_0^n.

The □ symbol (or [[1]]) separates ε_0-spaces within double square brackets just as the comma separates 0-spaces (single entries). In order to extend further, I nest higher-order double square bracket separators within double square bracket separators, starting with [[1 [[2]] 2]].

{a, b [[1 [[2]] 2]] 2} = {a ‹‹0 [[2]] 2›› b}
= {a ‹‹b ‹‹1›› b›› b}
= {a ‹‹ S □ S □ S □ ... □ S ›› b} (with b S strings),

where S = ‘b ‹‹0›› b’
= ‘b ‹b ‹ ... ‹b ‹b› b› ... › b› b’ (with b b’s from centre out).

{a, b [[1 [[3]] 2]] 2} = {a ‹‹0 [[3]] 2›› b}
= {a ‹‹b ‹‹2›› b›› b}
= {a ‹‹ S□S□...□S [[2]] S□S□...□S [[2]] ...... [[2]] S□S□...□S ›› b}
(with b^2 array of S’s within double angle brackets).

[[1 [[2]] 2]] has level ε_0^ω,
[[1 [[3]] 2]] has level ε_0^ω^2,
[[1 [[1, 2]] 2]] has level ε_0^ω^ω,
[[1 [[1, 1, 2]] 2]] has level ε_0^ω^ω^2,
[[1 [[1 [2] 2]] 2]] has level ε_0^ω^ω^ω,

[[1 [[1 [3] 2]] 2]] has level ε_0^ω^ω^ω^2,
[[1 [[1 [1, 2] 2]] 2]] has level ε_0^ω^ω^ω^ω,
[[1 [[X]] 2]] has level ε_0^ω^α.

[[1 [[1 □ 2]] 2]] has level ε_0^ε_0,
[[X [[1 □ 2]] 2]] has level ε_0^ε_0+α,
[[1 [[1 □ 2]] 3]] has level (ε_0^ε_0)2,
[[1 [[1 □ 2]] 1, 2]] has level (ε_0^ε_0)ω,
[[1 [[1 □ 2]] X]] has level (ε_0^ε_0)α,
[[1 [[1 □ 2]] 1 [[1 □ 2]] 2]] has level (ε_0^ε_0)^2,
[[1 [[2 □ 2]] 2]] has level (ε_0^ε_0)^ω,
[[1 [[1, 2 □ 2]] 2]] has level (ε_0^ε_0)^ω^ω,
[[1 [[X □ 2]] 2]] has level (ε_0^ε_0)^ω^α,
[[1 [[1 □ 3]] 2]] has level (ε_0^ε_0)^ε_0 = ε_0^ε_0^2,
[[1 [[1 □ 4]] 2]] has level ε_0^ε_0^3,
[[1 [[1 □ 1, 2]] 2]] has level ε_0^ε_0^ω,
[[1 [[1 □ 1 [2] 2]] 2]] has level ε_0^ε_0^ω^ω,
[[1 [[1 □ X]] 2]] has level ε_0^ε_0^α.

[[1 [[1 □ 1 □ 2]] 2]] has level ε_0^ε_0^ε_0,
[[1 [[1 □ 2 □ 2]] 2]] has level (ε_0^ε_0^ε_0)^ε_0 = ε_0^ε_0^(ε_0+1),
[[1 [[1 □ X □ 2]] 2]] has level ε_0^ε_0^(ε_0+α),
[[1 [[1 □ 1 □ 3]] 2]] has level ε_0^ε_0^($\varepsilon_0$2),
[[1 [[1 □ 1 □ X]] 2]] has level ε_0^ε_0^(ε_0α),
[[1 [[1 □ 1 □ 1 □ 2]] 2]] has level ε_0^ε_0^ε_0^2,
[[1 [[1 □ 1 □ X □ 2]] 2]] has level ε_0^ε_0^(ε_0^2+ε_0α),
[[1 [[1 □ 1 □ 1 □ 3]] 2]] has level ε_0^ε_0^((ε_0^2)2),
[[1 [[1 □ 1 □ 1 □ X]] 2]] has level ε_0^ε_0^((ε_0^2)α),
[[1 [[1 □ 1 □ 1 □ 1 □ 2]] 2]] has level ε_0^ε_0^ε_0^3,
[[1 [[1 [[2]] 2]] 2]] has level ε_0^ε_0^ε_0^ω,
[[1 [[1 [[1, 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ω^ω,
[[1 [[1 [[X]] 2]] 2]] has level ε_0^ε_0^ε_0^ω^α.

[[1 [[1 [[1 □ 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0,
[[X [[1 [[1 □ 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0+α,
[[1 [[1 [[1 □ 2]] 2]] 3]] has level (ε_0^ε_0^ε_0^ε_0)2,
[[1 [[1 [[1 □ 2]] 2]] X]] has level (ε_0^ε_0^ε_0^ε_0)α,
[[1 [[1 [[1 □ 2]] 2]] 1 [[1 [[1 □ 2]] 2]] 2]] has level (ε_0^ε_0^ε_0^ε_0)^2,
[[1 [[2 [[1 □ 2]] 2]] 2]] has level (ε_0^ε_0^ε_0^ε_0)^ω,
[[1 [[X [[1 □ 2]] 2]] 2]] has level (ε_0^ε_0^ε_0^ε_0)^ω^α,
[[1 [[1 □ 2 [[1 □ 2]] 2]] 2]] has level (ε_0^ε_0^ε_0^ε_0)^ε_0 = ε_0^ε_0^(ε_0^ε_0+1),
[[1 [[1 □ 1 □ 2 [[1 □ 2]] 2]] 2]] has level ε_0^ε_0^(ε_0^ε_0+ε_0),
[[1 [[1 [[2]] 2 [[1 □ 2]] 2]] 2]] has level ε_0^ε_0^(ε_0^ε_0+ε_0^ω),
[[1 [[1 [[1 □ 2]] 3]] 2]] has level ε_0^ε_0^((ε_0^ε_0)2),
[[1 [[1 [[1 □ 2]] X]] 2]] has level ε_0^ε_0^((ε_0^ε_0)α),
[[1 [[1 [[1 □ 2]] 1 □ 2]] 2]] has level ε_0^ε_0^((ε_0^ε_0)ε_0) = ε_0^ε_0^ε_0^(ε_0+1),
[[1 [[1 [[1 □ 2]] 1 [[2]] 2]] 2]] has level ε_0^ε_0^ε_0^(ε_0+ω),
[[1 [[1 [[1 □ 2]] 1 [[1 □ 2]] 2]] 2]] has level ε_0^ε_0^ε_0^($\varepsilon_0$2),
[[1 [[1 [[2 □ 2]] 2]] 2]] has level ε_0^ε_0^ε_0^(ε_0ω),

[[1 [[1 [[X □ 2]] 2]] 2]] has level ε_0^ε_0^ε_0^(ε_0(ω^α)),
[[1 [[1 [[1 □ 3]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^2,
[[1 [[1 [[1 □ 1, 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^ω,
[[1 [[1 [[1 □ X]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^α.

[[1 [[1 [[1 □ 1 □ 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^ε_0,
[[1 [[1 [[1 □ 1 □ X]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^(ε_0α),
[[1 [[1 [[1 □ 1 □ 1 □ 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^ε_0^2,
[[1 [[1 [[1 [[2]] 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^ε_0^ω,
[[1 [[1 [[1 [[X]] 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^ε_0^ω^α,
[[1 [[1 [[1 [[1 □ 2]] 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^ε_0^ε_0.

Take the separator
[[S_n]] = [[1 [[1 [[... [[1 [[1 □ 2]] 2]] ...]] 2]] 2]],
where there are n double squared brackets (ignoring the [[1]] in the centre, still symbolised by □), thus
[[S_1]] = [[1 □ 2]] has level $\varepsilon_0$2,
[[S_2]] = [[1 [[1 □ 2]] 2]] has level ε_0^ε_0,
[[S_3]] = [[1 [[1 [[1 □ 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0,
[[S_4]] = [[1 [[1 [[1 [[1 □ 2]] 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^ε_0^ε_0.

I conjecture that [[S_n]] has level ε_0^^(2n-2) for all positive integers of n ≥ 2. It holds true for n = 2, 3, 4.

Assuming it holds true for n = k,
[[S_k]] = [[1 [[S_{k-1}]] 2]] has level ε_0^^(2k-2),
[[X [[S_{k-1}]] 2]] has level ε_0^^(2k-2) + α,
[[1 [[S_{k-1}]] 3]] has level (ε_0^^(2k-2))2,
[[1 [[S_{k-1}]] X]] has level (ε_0^^(2k-2))α,
[[1 [[S_{k-1}]] 1 [[S_{k-1}]] 2]] has level (ε_0^^(2k-2))^2,
[[1 [[2 [[S_{k-2}]] 2]] 2]] has level (ε_0^^(2k-2))^ω,
[[1 [[X [[S_{k-2}]] 2]] 2]] has level (ε_0^^(2k-2))^ω^α,
[[1 [[1 □ 2 [[S_{k-2}]] 2]] 2]] has level (ε_0^^(2k-2))^ε_0 = ε_0^ε_0^(ε_0^^(2k-4) + 1),
[[1 [[1 [[S_m]] 2 [[S_{k-2}]] 2]] 2]] has level ε_0^ε_0^(ε_0^^(2k-4) + ε_0^^(2m)) (1 ≤ m ≤ k-3),
[[1 [[1 [[S_{k-2}]] 3]] 2]] has level ε_0^ε_0^((ε_0^^(2k-4))2),
[[1 [[1 [[S_{k-2}]] X]] 2]] has level ε_0^ε_0^((ε_0^^(2k-4))α),
[[1 [[1 [[S_{k-2}]] 1 [[S_m]] 2]] 2]] has level ε_0^ε_0^((ε_0^^(2k-4))(ε_0^^(2m))) (1 ≤ m ≤ k-3),
[[1 [[1 [[S_{k-2}]] 1 [[S_{k-2}]] 2]] 2]] has level ε_0^ε_0^((ε_0^^(2k-4))^2),
[[1 [[1 [[2 [[S_{k-3}]] 2]] 2]] 2]] has level ε_0^ε_0^((ε_0^^(2k-4))^ω),
[[1 [[1 [[X [[S_{k-3}]] 2]] 2]] 2]] has level ε_0^ε_0^((ε_0^^(2k-4))^ω^α),
[[1 [[1 [[1 □ 2 [[S_{k-3}]] 2]] 2]] 2]] has level ε_0^ε_0^((ε_0^^(2k-4))^ε_0) = ε_0^ε_0^ε_0^ε_0^(ε_0^^(2k-6) + 1),
[[1 [[1 [[1 [[S_m]] 2 [[S_{k-3}]] 2]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^(ε_0^^(2k-6) + ε_0^^(2m))
(1 ≤ m ≤ k-4),
[[1 [[1 [[1 [[S_{k-3}]] 3]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^((ε_0^^(2k-6))2),
[[1 [[1 [[1 [[S_{k-3}]] X]] 2]] 2]] has level ε_0^ε_0^ε_0^ε_0^((ε_0^^(2k-6))α),
......
[[1 [[1 [[... [[1 [[S_1]] X]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^((ε_0^^2)α)
(with k double squared brackets and 2k-4 ε_0's in 'ε_0^ε_0^...^ε_0'),
[[1 [[1 [[... [[1 [[S_1]] 1 [[S_1]] 2]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^((ε_0^ε_0)^2)
= ε_0^ε_0^...^ε_0^ε_0^($\varepsilon_0$2) (with 2k-2 ε_0's),
[[1 [[1 [[... [[1 [[2 □ 2]] 2]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^ε_0^(ε_0ω),

[[1 [[1 [[... [[1 [[X □ 2]] 2]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^ε_0^(ε_0(ω^α)),
[[1 [[1 [[... [[1 [[1 □ 3]] 2]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^ε_0^ε_0^2,
[[1 [[1 [[... [[1 [[1 □ X]] 2]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^ε_0^ε_0^α,
[[1 [[1 [[... [[1 [[1 □ 1 □ 2]] 2]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^ε_0^ε_0^ε_0 = ε_0^^(2k-1),
[[1 [[1 [[... [[1 [[1 □ 1 □ X]] 2]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^ε_0^ε_0^(ε_0α),
[[1 [[1 [[... [[1 [[1 □ 1 □ 1 □ 2]] 2]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^ε_0^ε_0^ε_0^2,
[[1 [[1 [[... [[1 [[1 [[2]] 2]] 2]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^ε_0^ε_0^ε_0^ω,
[[1 [[1 [[... [[1 [[1 [[X]] 2]] 2]] ...]] 2]] 2]] has level ε_0^ε_0^...^ε_0^ε_0^ε_0^ε_0^ω^α,
[[1 [[1 [[... [[1 [[1 [[1 □ 2]] 2]] 2]] ...]] 2]] 2]] = [[S_{k+1}]] has level ε_0^^(2k),

and so, it also holds true for n = k+1.

So, [[S_n]] has level ε_0^^(2n-2) for all positive integers of n ≥ 2 (outline of proof by induction shown above). Each time I add a layer of double square brackets to the separator the ε_0 power tower grows in height by two. When I introduce the next 'superlevel' of separators (starting with [[[1]]]) and define

{a, b [[[1]]] 2} = {a ‹‹‹0››› b}
= {a ‹‹b ‹‹b ‹‹ ... ‹‹b ‹‹b›› b›› ... ›› b›› b›› b}   (with b b's from centre to right),

which is equivalent to

{a, b [[1 [[1 [[... [[1 [[1, 2]] 2]] ...]] 2]] 2]] 2}   (with b-1 double square brackets, for b ≥ 2),

I would obtain an ε_1-recursive function. [[[1]]] separates ε_1-spaces and has level ε_1.

In fact, when I replace all of the double square brackets by treble square brackets in the above separators (with the □ symbol used in place of [[[1]]]), the ε_0's in the separator levels get replaced by ε_1's and I would obtain an ε_2-recursive function, using the [[[[1]]]] separator. (The proof that [[[S_n]]] has level ε_1^^(2n-2) for all n ≥ 2 would hold.) I can even continue this so that each additional square bracket around the separators adds one to the epsilon numbers, for example,

[[[...[1]...]]] (with n+2 square brackets around 1) would have level ε_n.

I now define the separator [X \ n] to be [[[...[X]...]]] (with n square brackets around the array X). Similarly, 'a ‹X \ n› b' is 'a ‹‹‹...‹X›...››› b' (with n angle brackets around the array X). By this definition

[1 \ n+2] has level ε_n.

The Main (M) Rules and Angle Bracket (A) Rules remain as before but a separator or angle bracket array can now end with a backslash (\) and a number (the 'superlevel' of the separator), when the number is 2 or greater. When this number is 1, the '\ 1' strings are omitted.

The only significant new rule to add would be:

{a, b [1 \ n] 2} = {a ‹0 \ n› b}
= {a ‹b ‹b ‹ ... ‹b ‹b \ n-1› b \ n-1› ... › b \ n-1› b \ n-1› b}
(with b b's from centre to right).

This is equivalent to

{a, b [1 [1 [... [1 [1, 2 \ n-1] 2 \ n-1] ...] 2 \ n-1] 2 \ n-1] 2} (with b-1 square brackets, for b ≥ 2)

and is an ε_{n-2}-recursive function, since [1 \ n] separates ε_{n-2}-spaces and has level ε_{n-2}.

The backslash is a 'hyperseparator', as it 'outranks' all the normal separators – even when they themselves contain the \ symbol. Furthermore, the backslash cannot appear in the 'base layer' of a main array (within curly brackets, as in {a, b \ c}). A number 1 between two separators is removed (along with separator to its left) when the left separator is of a lower level that the separator to its right (see Rules M2 and A2 of Nested Array Notation). The levels of the separators are determined by their associated ordinals, for example, [1 \ 3] ranks higher than [1 [2 \ 2] 2 \ 2] since they have ordinal levels of ε_1 and ε_0^ω respectively, and ε_1 > ε_0^ω. When the separator to the right of a number 1 is a ‘hyperseparator’ (represented by a backslash) and the 1 is not the first entry of its separator or angle bracket array (above the ‘base layer’ of the main array), the 1 is always removed along with the normal separator to its left. The following line is therefore added to Rule A2:

‘a ‹# [A] 1 \ n› b’ = ‘a ‹# \ n› b’.

In that line, [A] is a normal separator and # denotes the remainder of the angle bracket array.

The next stage of my extended Nested Array Notation would be to turn the number to the right of the backslash into an array.

{a, b [1 \ 1, 2] 2} = {a ‹0 \ 1, 2› b}
= {a ‹b \ b› b},

which is greater than

{a ‹0 \ b› b} = {a ‹b ‹b ‹ ... ‹b ‹b \ b-1› b \ b-1› ... › b \ b-1› b \ b-1› b}
(with b-1 angle brackets).

This is an ε_ω-recursive function, since [1 \ 1, 2] separates ε_ω-spaces and has level ε_ω.

Angle Bracket Rule A4 with k = 2, n = 2 and # blank is

‘a ‹0 [A_1] 1 [A_2] 2› b’ = ‘a ‹ b ‹A_1’› b [A_1] b ‹A_2’› b › b’.

When [A_1] is a backslash and [A_2] is a comma, the strings A_1’ and A_2’ are both set to ‘0’, since they are at the very first level of hyperseparator and separator respectively. Thus

‘a ‹0 \ 1, 2› b’ = ‘a ‹ b ‹0› b \ b ‹0› b › b’
= ‘a ‹b \ b› b’.

For example,

{3, 2 [1 \ 1, 2] 2} = {3 ‹0 \ 1, 2› 2}
= {3 ‹2 \ 2› 2}
= {3 ‹0\2› 2 [1\2] 3 ‹0\2› 2 [2\2] 3 ‹0\2› 2 [1\2] 3 ‹0\2› 2}
= {3,3 [2] 3,3 [1\2] 3,3 [2] 3,3 [2\2] 3,3 [2] 3,3 [1\2] 3,3 [2] 3,3},
{3, 2, 2 [1 \ 1, 2] 2} = {3, 3 [1 \ 1, 2] 2}
= {3 ‹0 \ 1, 2› 3}
= {3 ‹3 \ 3› 3}
= N,
{3, 3, 2 [1 \ 1, 2] 2} = {3, N [1 \ 1, 2] 2}
= {3 ‹0 \ 1, 2› N}
= {3 ‹N \ N› N}.

In Rule A2, we now have

‘a ‹# [A] 1 \ #*› b’ = ‘a ‹# \ #*› b’

instead of

‘a ‹# [A] 1 \ n› b’ = ‘a ‹# \ n› b’.

#* denotes the remainder of the angle bracket array to the right of the backslash.

When I replace all of the double square brackets in the list of separators with double square brackets in such a way so that each [[S]] is replaced by [S \ 1, 2] (with the □ symbol used in place of [1 \ 1, 2]), the ε_0’s in the separator levels get replaced by ε_ω’s and I would obtain an $\varepsilon_{\omega+1}$-recursive function, using the [1 \ 2, 2] separator, which has level $\varepsilon_{\omega+1}$. I can repeat this process when introducing higher level separators, so that if [1 \ n #] has level ε_α (where # is rest of array) then [1 \ n+1 #] would have level $\varepsilon_{\alpha+1}$.

The following separators have ordinal levels as follows:-

[1 \ 1, 2] has level ε_{ω},
[1 \ 2, 2] has level $\varepsilon_{\omega+1}$,
[1 \ 3, 2] has level $\varepsilon_{\omega+2}$,
[1 \ 1, 3] has level $\varepsilon_{\omega 2}$,
[1 \ 2, 3] has level $\varepsilon_{\omega 2+1}$,
[1 \ 1, 4] has level $\varepsilon_{\omega 3}$,
[1 \ 1, 1, 2] has level $\varepsilon_{\omega\text{^}2}$,
[1 \ 1, 1, 1, 2] has level $\varepsilon_{\omega\text{^}3}$,
[1 \ 1 [2] 2] has level $\varepsilon_{\omega\text{^}\omega}$,
[1 \ 1 [3] 2] has level $\varepsilon_{\omega\text{^}\omega\text{^}2}$,
[1 \ 1 [4] 2] has level $\varepsilon_{\omega\text{^}\omega\text{^}3}$,
[1 \ 1 [1, 2] 2] has level $\varepsilon_{\omega\text{^}\omega\text{^}\omega}$,
[1 \ 1 [1 [2] 2] 2] has level $\varepsilon_{\omega\text{^}\omega\text{^}\omega\text{^}\omega}$,
[1 \ 1 [1 [1, 2] 2] 2] has level $\varepsilon_{\omega\text{^}\omega\text{^}\omega\text{^}\omega\text{^}\omega}$.

[1 \ 1 [1 \ 2] 2] has level $\varepsilon(\varepsilon_0)$,
[1 \ 2 [1 \ 2] 2] has level $\varepsilon(\varepsilon_0+1)$,
[1 \ 1, 2 [1 \ 2] 2] has level $\varepsilon(\varepsilon_0+\omega)$,
[1 \ 1 [2] 2 [1 \ 2] 2] has level $\varepsilon(\varepsilon_0+\omega\text{^}\omega)$,
[1 \ 1 [1 \ 2] 3] has level $\varepsilon(\varepsilon_0 2)$,
[1 \ 1 [1 \ 2] 1 [1 \ 2] 2] has level $\varepsilon(\varepsilon_0\text{^}2)$,
[1 \ 1 [2 \ 2] 2] has level $\varepsilon(\varepsilon_0\text{^}\omega)$,
[1 \ 1 [1, 2 \ 2] 2] has level $\varepsilon(\varepsilon_0\text{^}\omega\text{^}\omega)$,
[1 \ 1 [1 [1 \ 2] 2 \ 2] 2] has level $\varepsilon(\varepsilon_0\text{^}\varepsilon_0)$,
[1 \ 1 [1 \ 3] 2] has level $\varepsilon(\varepsilon_1)$,
[1 \ 1 [1 \ 4] 2] has level $\varepsilon(\varepsilon_2)$,
[1 \ 1 [1 \ 1, 2] 2] has level $\varepsilon(\varepsilon_{\omega})$,
[1 \ 1 [1 \ 2, 2] 2] has level $\varepsilon(\varepsilon_{\omega+1})$,
[1 \ 1 [1 \ 1, 3] 2] has level $\varepsilon(\varepsilon_{\omega 2})$,
[1 \ 1 [1 \ 1, 1, 2] 2] has level $\varepsilon(\varepsilon_{\omega\text{^}2})$,
[1 \ 1 [1 \ 1 [2] 2] 2] has level $\varepsilon(\varepsilon_{\omega\text{^}\omega})$,
[1 \ 1 [1 \ 1 [1, 2] 2] 2] has level $\varepsilon(\varepsilon_{\omega\text{^}\omega\text{^}\omega})$.

[1 \ 1 [1 \ 1 [1 \ 2] 2] 2] has level $\varepsilon(\varepsilon(\varepsilon_0))$,
[1 \ 1 [1 \ 1 [1 \ 3] 2] 2] has level $\varepsilon(\varepsilon(\varepsilon_1))$,
[1 \ 1 [1 \ 1 [1 \ 1, 2] 2] 2] has level $\varepsilon(\varepsilon(\varepsilon_{\omega}))$,
[1 \ 1 [1 \ 1 [1 \ 1 [2] 2] 2] 2] has level $\varepsilon(\varepsilon(\varepsilon_{\omega\text{^}\omega}))$,
[1 \ 1 [1 \ 1 [1 \ 1 [1 \ 2] 2] 2] 2] has level $\varepsilon(\varepsilon(\varepsilon(\varepsilon_0)))$,
[1 \ 1 [1 \ 1 [1 \ 1 [1 \ 1 [1 \ 2] 2] 2] 2] 2] has level $\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon_0))))$.

When the array to the left of the backslash is a single 1, the array to the right of the backslash changes much like simple nested separator arrays (without the backslash).

When n ≥ 2 and # is rest of array,
{a, b [1 \ n #] 2} = {a ‹0 \ n #› b}
= {a ‹b ‹b ‹ ... ‹b ‹b \ n-1 #› b \ n-1 #› ... › b \ n-1 #› b \ n-1 #› b}
(with b b’s from centre to right).

With k 1’s to right of backslash,

{a, b [1 \ 1, 1, ... 1, n #] 2} = {a ‹0 \ 1, 1, ... 1, n #› b}
= {a ‹b \ b, b, ... , b, n-1 #› b}
(with k b’s between backslash and closed angle bracket).

{a, b [1 \ 1 [2] n #] 2} = {a ‹0 \ 1 [2] n #› b}
= {a ‹b \ b ‹1› b [2] n-1 #› b}
= {a ‹b \ b, b, ... , b [2] n-1 #› b} (with b b’s between \ and [2]).

{a, b [1 \ 1 [1, 2] n #] 2} = {a ‹0 \ 1 [1, 2] n #› b}
= {a ‹b \ b ‹0, 2› b [1, 2] n-1 #› b}
= {a ‹b \ b ‹b› b [1, 2] n-1 #› b}
(b^b array of b’s between \ and [1, 2]).

When X is a string of characters,

{a, b [1 \ 1 [X] n #] 2} = {a ‹0 \ 1 [X] n #› b}
= {a ‹b \ b ‹X’› b [X] n-1 #› b},

where X’ is identical to X except that the first entry has been reduced by 1. If X begins with 1, X’ begins with 0. This bears similarities with Rule A4.

{a, b [1 \ 1 [1 \ 2] 2] 2} = {a ‹0 \ 1 [1 \ 2] 2› b}
= {a ‹b \ b ‹0 \ 2› b› b},
= {a ‹b \ S› b},

where S = ‘b ‹b ‹b ‹ ... ‹b ‹b› b› ... › b› b› b’ (with b b’s from centre to right).

{a, b [1 \ 1 [1 \ n #] 2] 2} = {a ‹0 \ 1 [1 \ n #] 2› b}
= {a ‹b \ b ‹0 \ n #› b› b}
= {a ‹b \ S› b},

where S = ‘b ‹b ‹b ‹ ... ‹b ‹b \ n-1 #› b \ n-1 #› ... › b \ n-1 #› b \ n-1 #› b’
(with b b’s from centre to right).

{a, b [1 \ 1 [1 \ 1 [1 \ 2] 2] 2] 2} = {a ‹0 \ 1 [1 \ 1 [1 \ 2] 2] 2› b}
= {a ‹b \ b ‹0 \ 1 [1 \ 2] 2› b› b}
= {a ‹b \ b ‹b \ b ‹0 \ 2› b› b› b}
= {a ‹b \ b ‹b \ S› b› b},

where S = ‘b ‹b ‹b ‹ ... ‹b ‹b› b› ... › b› b› b’ (with b b’s from centre to right).

I next define the lowest separator with more than one backslash,

{a, b [1 \ 1 \ 2] 2} = {a ‹0 \ 1 \ 2› b}
= {a ‹b \ b ‹b \ b ‹ ... ‹b \ b ‹b \ b› b› ... › b› b› b} (with b-1 angle brackets).

This is equivalent to

{a, b [1 \ 1 [1 \ 1 [... [1 \ 1 [1 \ 1, 2] 2] ...] 2] 2] 2} (with b-1 square brackets, for b ≥ 2)

and is a ζ_0-recursive function, since [1 \ 1 \ 2] separates ζ_0-spaces and has level

$\zeta_0 = \varepsilon(\varepsilon(\varepsilon(...(\varepsilon_0)...)))$ (with ω epsilons).

ζ_0 (ζ for zeta) is the first ordinal beyond the epsilons (φ(2, 0) in the Veblen hierarchy).

The most significant single backslashed separators are:-

[1 \ 2] has level ε_0,

[2 \ 2] has level ε_0+1,

$[1\ [1 \backslash 2]\ 2 \backslash 2]$ has level $\varepsilon_0 2$,
$[1\ [2 \backslash 2]\ 2 \backslash 2]$ has level ε_0^ω,
$[1\ [1\ [1 \backslash 2]\ 2 \backslash 2]\ 2 \backslash 2]$ has level ε_0^ε_0,
$[1 \backslash 3]$ has level ε_1,
$[1 \backslash 4]$ has level ε_2,
$[1 \backslash 1, 2]$ has level ε_ω,
$[1 \backslash 1\ [1 \backslash 2]\ 2]$ has level $\varepsilon(\varepsilon_0)$,
$[1 \backslash 1\ [1 \backslash 1\ [1 \backslash 2]\ 2]\ 2]$ has level $\varepsilon(\varepsilon(\varepsilon_0))$.

Here are some separators with two backslashes:-

$[1 \backslash 1 \backslash 2]$ has level ζ_0,
$[1\ [1 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0+\varepsilon_0$,
$[1\ [1 \backslash 2]\ 1\ [1 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0+\varepsilon_0$^2,
$[1\ [2 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0+\varepsilon_0$^ω,
$[1\ [1\ [1 \backslash 2]\ 2 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0+\varepsilon_0$^ε_0,
$[1\ [1\ [1 \backslash 2]\ 1\ [1 \backslash 2]\ 2 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0+\varepsilon_0$^ε_0^ε_0,
$[1\ [1\ [1\ [1 \backslash 2]\ 2 \backslash 2]\ 2 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0+\varepsilon_0$^ε_0^ε_0^ε_0,
$[1\ [1 \backslash 3]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0+\varepsilon_1$,
$[1\ [1 \backslash 1, 2]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0+\varepsilon_\omega$,
$[1\ [1 \backslash 1\ [1 \backslash 2]\ 2]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0+\varepsilon(\varepsilon_0)$,
$[1\ [1 \backslash 1\ [1 \backslash 1\ [1 \backslash 2]\ 2]\ 2]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0+\varepsilon(\varepsilon(\varepsilon_0))$.

$[1\ [1 \backslash 1 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level $\zeta_0 2$,
$[1\ [2 \backslash 1 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level ζ_0^ω,
$[1\ [1\ [1 \backslash 2]\ 2 \backslash 1 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level ζ_0^ε_0,
$[1\ [1\ [1 \backslash 1\ [1 \backslash 2]\ 2]\ 2 \backslash 1 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level ζ_0^$\varepsilon(\varepsilon_0)$,
$[1\ [1\ [1 \backslash 1 \backslash 2]\ 2 \backslash 1 \backslash 2]\ 2 \backslash 1 \backslash 2]$ has level ζ_0^ζ_0.

Since ζ_0 is a fixed point of $\alpha = \varepsilon_\alpha$, it follows that $\zeta_0 = \varepsilon(\zeta_0)$, and so, ζ_0^^$\omega = \varepsilon(\zeta_0)$^^$\omega = \varepsilon(\zeta_0+1)$. We need $\varepsilon(\varepsilon(...\varepsilon(\zeta_0+1)...))$ (with ω ε's) to make ζ_1. The ordinal $\zeta_{\alpha+1}$ follows on from ζ_α in a similar manner.

$[1 \backslash 2 \backslash 2]$ has level $\varepsilon(\zeta_0+1) = \zeta_0$^^$\omega$,
$[1 \backslash 3 \backslash 2]$ has level $\varepsilon(\zeta_0+2) = \varepsilon(\zeta_0+1)$^^$\omega$,
$[1 \backslash 1, 2 \backslash 2]$ has level $\varepsilon(\zeta_0+\omega)$,
$[1 \backslash 1\ [1 \backslash 2]\ 2 \backslash 2]$ has level $\varepsilon(\zeta_0+\varepsilon_0)$,
$[1 \backslash 1\ [1 \backslash 1 \backslash 2]\ 2 \backslash 2]$ has level $\varepsilon(\zeta_0 2)$,
$[1 \backslash 1\ [1 \backslash 1\ [1 \backslash 1 \backslash 2]\ 2 \backslash 2]\ 2 \backslash 2]$ has level $\varepsilon(\varepsilon(\zeta_0 2))$,
$[1 \backslash 1\ [1 \backslash 1\ [1 \backslash 1\ [1 \backslash 1 \backslash 2]\ 2 \backslash 2]\ 2 \backslash 2]\ 2 \backslash 2]$ has level $\varepsilon(\varepsilon(\varepsilon(\zeta_0 2)))$.

$[1 \backslash 1 \backslash 3]$ has level ζ_1,
$[1 \backslash 2 \backslash 3]$ has level $\varepsilon(\zeta_1+1)$,
$[1 \backslash 1, 2 \backslash 3]$ has level $\varepsilon(\zeta_1+\omega)$,
$[1 \backslash 1\ [1 \backslash 2]\ 2 \backslash 3]$ has level $\varepsilon(\zeta_1+\varepsilon_0)$,
$[1 \backslash 1\ [1 \backslash 1 \backslash 2]\ 2 \backslash 3]$ has level $\varepsilon(\zeta_1+\zeta_0)$,
$[1 \backslash 1\ [1 \backslash 1 \backslash 3]\ 2 \backslash 3]$ has level $\varepsilon(\zeta_1 2)$,
$[1 \backslash 1\ [1 \backslash 1\ [1 \backslash 1 \backslash 3]\ 2 \backslash 3]\ 2 \backslash 3]$ has level $\varepsilon(\varepsilon(\zeta_1 2))$.

$[1 \backslash 1 \backslash 4]$ has level ζ_2,

[1 \ 1 \ 5] has level ζ_3,
[1 \ 1 \ 1, 2] has level ζ_ω,
[1 \ 1 \ 1 [1 \ 2] 2] has level $\zeta(\varepsilon_0)$,
[1 \ 1 \ 1 [1 \ 1 \ 2] 2] has level $\zeta(\zeta_0)$,
[1 \ 1 \ 1 [1 \ 1 \ 1 [1 \ 1 \ 2] 2] 2] has level $\zeta(\zeta(\zeta_0))$,
[1 \ 1 \ 1 [1 \ 1 \ 1 [1 \ 1 \ 1 [1 \ 1 \ 2] 2] 2] 2] has level $\zeta(\zeta(\zeta(\zeta_0)))$.

The sequence of separators starting with the last three has limit ordinal φ(3, 0). Note that $\varphi(1, \alpha) = \varepsilon_\alpha$ and $\varphi(2, \alpha) = \zeta_\alpha$.

{a, b [1 \ 1 \ 1 \ 2] 2} = {a ‹0 \ 1 \ 1 \ 2› b}
= {a ‹b\b\ b ‹b\b\ b ‹ ... ‹b\b\ b ‹b\b\b› b› ... › b› b› b}
(with b-1 angle brackets)

is a φ(3, 0)-recursive function, since
[1 \ 1 \ 1 \ 2] has level φ(3, 0).

[1 \ 2 \ 1 \ 2] has level $\varepsilon_{\varphi(3,\,0)+1}$ = φ(3, 0)^^ω,
[1 \ 3 \ 1 \ 2] has level $\varepsilon_{\varphi(3,\,0)+2}$,
[1 \ 1 [1 \ 1 \ 1 \ 2] 2 \ 1 \ 2] has level $\varepsilon_{\varphi(3,\,0)2}$,
[1 \ 1 \ 2 \ 2] has level $\zeta_{\varphi(3,\,0)+1} = \varepsilon(\varepsilon(...(\varepsilon_{\varphi(3,\,0)2})...))$ (with ω ε's),
[1 \ 1 \ 3 \ 2] has level $\zeta_{\varphi(3,\,0)+2}$,
[1 \ 1 \ 1 [1 \ 1 \ 1 \ 2] 2 \ 2] has level $\zeta_{\varphi(3,\,0)2}$,
[1 \ 1 \ 1 \ 3] has level $\varphi(3, 1) = \zeta(\zeta(...(\zeta_{\varphi(3,\,0)2})...))$ (with ω ζ's),
[1 \ 1 \ 1 \ 4] has level φ(3, 2),
[1 \ 1 \ 1 \ 1 [1 \ 1 \ 1 \ 2] 2] has level φ(3, φ(3, 0)),
[1 \ 1 \ 1 \ 1 [1 \ 1 \ 1 \ 1 [1 \ 1 \ 1 \ 2] 2] 2] has level φ(3, φ(3, φ(3, 0))).

[1 \ 1 \ 1 \ 1 \ 2] has level φ(4, 0),
[1 \ 1 \ 1 \ 1 \ 3] has level φ(4, 1),
[1 \ 1 \ 1 \ 1 \ 1 [1 \ 1 \ 1 \ 1 \ 2] 2] has level φ(4, φ(4, 0)),
[1 \ 1 \ 1 \ 1 \ 1 [1 \ 1 \ 1 \ 1 \ 1 [1 \ 1 \ 1 \ 1 \ 2] 2] 2] has level φ(4, φ(4, φ(4, 0))),
[1 \ 1 \ 1 \ 1 \ 1 \ 2] has level φ(5, 0),
[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2] has level φ(6, 0),
[1 \ 1 \ 1 \ ... \ 1 \ 2] (with n 1's) has level φ(n, 0),
[1 \ 1 \ 1 \ ... \ 1 \ m] (with n 1's) has level φ(n, m-2).

Suppose that X is a string of characters such that the separator [X] has level α (a combination of a finite number and the various ordinal infinities). For example,
if [X] is a single entry of a finite number, say, [a], then α = a-1;
if [X] = [a, b], then α = ω(b-1) + a-1;
if [X] = [1 [a] 2], then α = ω^ω^(a-1);
if [X] = [1 [1 \ a] 2], then $\alpha = \varepsilon_{a-2}$.

The effect of altering the entries of
[S] = [1 \ 1 \ 1 \ ... \ 1 \ m] (with n 1's)
is now considered. This separator has level φ(n, m-2).

If the 1st 1 of [S] is replaced by X, [S] would have level φ(n, m-2)+α;
if the 2nd 1 of [S] is replaced by X, [S] would have level $\varepsilon_{\varphi(n,\,m-2)+\alpha}$;
if the 3rd 1 of [S] is replaced by X, [S] would have level $\zeta_{\varphi(n,\,m-2)+\alpha}$;

if the 4th 1 of [S] is replaced by X, [S] would have level φ(3, φ(n, m-2)+α);
if the kth 1 of [S] is replaced by X, [S] would have level φ(k-1, φ(n, m-2)+α).

If the nth 1 of [S] is replaced by X, [S] would have level φ(n-1, φ(n, m-2)+α); the limit ordinal of repeatedly replacing the nth 1 of [S] by '1 [S] 2' would be φ(n, m-1), the level of the separator

[1 \ 1 \ 1 \ ... \ 1 \ m+1]   (with n 1's).

If the m of [S] is replaced by '1 [S] 2' with m = 2, [S] would have level φ(n, φ(n, 0)); the limit ordinal of repeatedly doing this would be φ(n+1, 0), the level of the separator

[1 \ 1 \ 1 \ ... \ 1 \ 1 \ 2]   (with n+1 1's).

In general, with a string of k 1's and backslashes in separator (k ≥ 1),

{a, b [1 \ 1 \ 1 \ ... \ 1 \ 2] 2} = {a ‹0 \ 1 \ 1 \ ... \ 1 \ 2› b}
= {a ‹R_b› b},

where R_n = 'b \ b \ ... \ b \ b ‹R_{n-1}› b'   (with k-1 backslashes, n > 1),
R_1 = '0'.

Note that

R_2 = 'b \ b \ ... \ b \ b'   (with k b's),

since 'b ‹R_1› b' = 'b ‹0› b' = 'b'.

For example, when k = 4,

{a, 2 [1 \ 1 \ 1 \ 1 \ 2] 2} = {a ‹0 \ 1 \ 1 \ 1 \ 2› 2}
= {a ‹2 \ 2 \ 2 \ 2› 2},
{a, 3 [1 \ 1 \ 1 \ 1 \ 2] 2} = {a ‹0 \ 1 \ 1 \ 1 \ 2› 3}
= {a ‹ 3\3\3\ 3 ‹3\3\3\3› 3 › 3},
{a, 4 [1 \ 1 \ 1 \ 1 \ 2] 2} = {a ‹0 \ 1 \ 1 \ 1 \ 2› 4}
= {a ‹ 4\4\4\ 4 ‹ 4\4\4\ 4 ‹4\4\4\4› 4 › 4 › 4}.

While simple nested separators (without the backslash symbol) are required to reach the ε_0 level separator [1 \ 2], the ε_1 level separator [1 \ 3] involves nested 'superlevel 2' brackets (with '\ 2' appended to them), the ε_2 separator [1 \ 4] involves nested 'superlevel 3' brackets, and so on. In order to reach the $\varepsilon(\varepsilon_0)$ level separator, the superlevels of the brackets would need to take on ε_0 level complexity themselves, and they would require $\varepsilon(\varepsilon_0)$ level complexity just to get to $\varepsilon(\varepsilon(\varepsilon_0))$. The ζ_0 or φ(2, 0) level separator [1 \ 1 \ 2] actually requires nested superlevels (arrays to the right of the backslash) or 'supernested' separators – in short, a Super-Nested Array Notation. While the $\varepsilon(\zeta_0+1)$ separator [1 \ 2 \ 2] involves nested 'super-superlevel 2' or '2-superlevel 2' brackets, the ζ_1 separator [1 \ 1 \ 3] needs supernested '2-superlevel 2' brackets. Reaching $\zeta(\zeta_0)$ requires the bracket 2-superlevels to be of ζ_0 level complexity and '2-supernested' separators are needed to achieve the φ(3, 0) level separator [1 \ 1 \ 1 \ 2]. Reaching φ(4, 0) requires '3-supernested' separators and φ(5, 0) involves '4-supernested' separators, and so on. The extended Super-Nested Array Notation, with an unlimited number of backslashes in the 'base layers' of separators, has a limit ordinal of φ(ω, 0).

In order to go further, I need to introduce the next 'hyperseparator' symbol – the double backslash (\\). As with the single backslash, it can only appear within a separator or angle bracket array. The lowest separator to contain this is [1 \\ 2], which is defined as follows:

{a, b [1 \\ 2] 2} = {a ‹0 \\ 2› b}
= {a ‹ b \ b \ b \ ... \ b › b}   (with b b's within angle brackets).

This is a φ(ω, 0)-recursive function.

For example,

{3, 2 [1 \\ 2] 2} = {3 ‹0 \\ 2› 2}
= {3 ‹2 \ 2› 2}
= {3 ‹0\2› 2 [1\2] 3 ‹0\2› 2 [2\2] 3 ‹0\2› 2 [1\2] 3 ‹0\2› 2}
= {3,3 [2] 3,3 [1\2] 3,3 [2] 3,3 [2\2] 3,3 [2] 3,3 [1\2] 3,3 [2] 3,3},
{3, 3 [1 \\ 2] 2} = {3 ‹0 \\ 2› 3}
= {3 ‹3 \ 3 \ 3› 3},
{3, 4 [1 \\ 2] 2} = {3 ‹0 \\ 2› 4}
= {3 ‹4 \ 4 \ 4 \ 4› 4}.

[1 \\ 2] has level φ(ω, 0),
[2 \\ 2] has level φ(ω, 0)+1,
[1 [1 \ 2] 2 \\ 2] has level φ(ω, 0)+ε_0,
[1 [1 \ 1 \ 1 \ ... \ 1 \ 2] 2 \\ 2] (with n 1's within inner brackets) has level φ(ω, 0)+φ(n, 0),
[1 [1 \\ 2] 2 \\ 2] has level φ(ω, 0)2,
[1 [1 \\ 2] 1 [1 \\ 2] 2 \\ 2] has level φ(ω, 0)^2,
[1 [2 \\ 2] 2 \\ 2] has level φ(ω, 0)^ω,
[1 [1 [1 \ 2] 2 \\ 2] 2 \\ 2] has level φ(ω, 0)^ε_0,
[1 [1 [1 \\ 2] 2 \\ 2] 2 \\ 2] has level φ(ω, 0)^φ(ω, 0),
[1 \ 2 \\ 2] has level $\varepsilon_{\varphi(\omega,\ 0)+1}$ = φ(ω, 0)^^ω,
[1 \ 1 \ 1 \ ... \ 1 \ 2 \\ 2] (with n 1's) has level φ(n, φ(ω, 0)+1).

I next define

{a, b [1 \\ n #] 2} = {a ‹0 \\ n #› b}
= {a ‹ b \ b \ b \ ... \ b \\ n-1 # › b} (with b b's within angle brackets),

where # is rest of array.

Since φ(n, φ(ω, α)+1) has limit ordinal φ(ω, α+1) as n tends to ω, it follows that

[1 \\ 3] has level φ(ω, 1),
[1 \ 1 \ 1 \ ... \ 1 \ 2 \\ 3] (with n 1's) has level φ(n, φ(ω, 1)+1),
[1 \\ 4] has level φ(ω, 2),
[1 \ 1 \ 1 \ ... \ 1 \ 2 \\ 4] (with n 1's) has level φ(n, φ(ω, 2)+1),
[1 \\ 5] has level φ(ω, 3),
[1 \\ 6] has level φ(ω, 4),
[1 \\ 1, 2] has level φ(ω, ω),
[1 \\ 2, 2] has level φ(ω, ω+1),
[1 \\ 3, 2] has level φ(ω, ω+2),
[1 \\ 1, 3] has level φ(ω, ω2),
[1 \\ 1, 4] has level φ(ω, ω3),
[1 \\ 1, 1, 2] has level φ(ω, ω^2),
[1 \\ 1, 1, 3] has level φ(ω, (ω^2)2),
[1 \\ 1, 1, 1, 2] has level φ(ω, ω^3),
[1 \\ 1, 1, 1, 1, 2] has level φ(ω, ω^4),
[1 \\ 1 [2] 2] has level φ(ω, ω^ω),
[1 \\ 1 [1 \ 2] 2] has level φ(ω, ε_0),
[1 \\ 1 [1 \\ 2] 2] has level φ(ω, φ(ω, 0)),
[1 \\ 1 [1 \\ 1 [1 \\ 2] 2] 2] has level φ(ω, φ(ω, φ(ω, 0))),
[1 \\ 1 [1 \\ 1 [1 \\ 1 [1 \\ 2] 2] 2] 2] has level φ(ω, φ(ω, φ(ω, φ(ω, 0)))).

[1 \\ 1 \ 2] has level φ(ω+1, 0) = φ(ω, φ(ω+1, 0)),
[1 \ 1 \ 1 \ ... \ 1 \ 2 \\ 1 \ 2] (with n 1's before \\) has level φ(n, φ(ω+1, 0)+1),
[1 \\ 2 \ 2] has level φ(ω, φ(ω+1, 0)+1),
[1 \\ 3 \ 2] has level φ(ω, φ(ω+1, 0)+2),
[1 \\ 1, 2 \ 2] has level φ(ω, φ(ω+1, 0)+ω),
[1 \\ 1, 3 \ 2] has level φ(ω, φ(ω+1, 0)+ω2),
[1 \\ 1, 1, 2 \ 2] has level φ(ω, φ(ω+1, 0)+ω^2),
[1 \\ 1, 1, 1, 2 \ 2] has level φ(ω, φ(ω+1, 0)+ω^3),
[1 \\ 1 [2] 2 \ 2] has level φ(ω, φ(ω+1, 0)+ω^ω),
[1 \\ 1 [1 \\ 1 \ 2] 2 \ 2] has level φ(ω, φ(ω+1, 0)2),
[1 \\ 1 [1 \\ 1 [1 \\ 1 \ 2] 2 \ 2] 2 \ 2] has level φ(ω, φ(ω, φ(ω+1, 0)2)),
[1 \\ 1 [1 \\ 1 [1 \\ 1 [1 \\ 1 \ 2] 2 \ 2] 2 \ 2] 2 \ 2] has level φ(ω, φ(ω, φ(ω, φ(ω+1, 0)2))),
[1 \\ 1 \ 3] has level φ(ω+1, 1) = φ(ω, φ(ω+1, 1)).

The development of the [1 \\ X \ 3] separators mirrors the [1 \\ X \ 2] set except that φ(ω+1, 0) is replaced by φ(ω+1, 1) in each of the associated ordinals. It follows that

[1 \\ 1 \ 4] has level φ(ω+1, 2),
[1 \\ 1 \ 5] has level φ(ω+1, 3),
[1 \\ 1 \ 1, 2] has level φ(ω+1, ω),
[1 \\ 1 \ 1, 3] has level φ(ω+1, ω2),
[1 \\ 1 \ 1, 1, 2] has level φ(ω+1, ω^2),
[1 \\ 1 \ 1, 1, 1, 2] has level φ(ω+1, ω^3),
[1 \\ 1 \ 1 [2] 2] has level φ(ω+1, ω^ω),
[1 \\ 1 \ 1 [1 \ 2] 2] has level φ(ω+1, ε_0),
[1 \\ 1 \ 1 [1 \\ 1 \ 2] 2] has level φ(ω+1, φ(ω+1, 0)),
[1 \\ 1 \ 1 [1 \\ 1 \ 1 [1 \\ 1 \ 2] 2] 2] has level φ(ω+1, φ(ω+1, φ(ω+1, 0))),
[1 \\ 1 \ 1 [1 \\ 1 \ 1 [1 \\ 1 \ 1 [1 \\ 1 \ 2] 2] 2] 2] has level φ(ω+1, φ(ω+1, φ(ω+1, φ(ω+1, 0)))),
[1 \\ 1 \ 1 \ 2] has level φ(ω+2, 0).

[1 \\ X \ 1 \ 2] mirrors the [1 \\ X \ 2] set except that φ(ω+1, 0) is replaced by φ(ω+1, φ(ω+2, 0)) in each of the associated ordinals. It follows that

[1 \\ 1 \ 2 \ 2] has level φ(ω+1, φ(ω+2, 0)+1),
[1 \\ 1 \ 3 \ 2] has level φ(ω+1, φ(ω+2, 0)+2),
[1 \\ 1 \ 1 [1 \\ 1 \ 1 \ 2] 2 \ 2] has level φ(ω+1, φ(ω+2, 0)2),
[1 \\ 1 \ 1 [1 \\ 1 \ 1 [1 \\ 1 \ 1 \ 2] 2 \ 2] 2 \ 2] has level φ(ω+1, φ(ω+1, φ(ω+2, 0)2)),
[1 \\ 1 \ 1 \ 3] has level φ(ω+2, 1),
[1 \\ 1 \ 1 \ 4] has level φ(ω+2, 2),
[1 \\ 1 \ 1 \ 1, 2] has level φ(ω+2, ω),
[1 \\ 1 \ 1 \ 1 [1 \\ 1 \ 1 \ 2] 2] has level φ(ω+2, φ(ω+2, 0)),
[1 \\ 1 \ 1 \ 1 [1 \\ 1 \ 1 \ 1 [1 \\ 1 \ 1 \ 2] 2] 2] has level φ(ω+2, φ(ω+2, φ(ω+2, 0))).

[1 \\ 1 \ 1 \ 1 \ 2] has level φ(ω+3, 0),
[1 \\ 1 \ 1 \ 1 \ 3] has level φ(ω+3, 1),
[1 \\ 1 \ 1 \ 1 \ 1 [1 \\ 1 \ 1 \ 1 \ 2] 2] has level φ(ω+3, φ(ω+3, 0)),
[1 \\ 1 \ 1 \ 1 \ 1 [1 \\ 1 \ 1 \ 1 \ 1 [1 \\ 1 \ 1 \ 1 \ 2] 2] 2] has level φ(ω+3, φ(ω+3, φ(ω+3, 0))),
[1 \\ 1 \ 1 \ 1 \ 1 \ 2] has level φ(ω+4, 0),
[1 \\ 1 \ 1 \ 1 \ 1 \ 1 \ 2] has level φ(ω+5, 0),
[1 \\ 1 \ 1 \ ... \ 1 \ 2] (with n 1's after \\) has level φ(ω+n, 0),
[1 \\ 1 \ 1 \ ... \ 1 \ m] (with n 1's after \\) has level φ(ω+n, m-2).

With two double backslashes,

[1 \\ 1 \\ 2] has level φ(ω2, 0),
[1 \ 2 \\ 1 \\ 2] has level $\varepsilon_{\varphi(\omega2,\ 0)+1}$ = φ(ω2, 0)^^ω,
[1 \ 1 \ ... \ 1 \ 2 \\ 1 \\ 2] (with n 1's before first \\) has level φ(n, φ(ω2, 0)+1),
[1 \\ 2 \\ 2] has level φ(ω, φ(ω2, 0)+1),
[1 \\ 3 \\ 2] has level φ(ω, φ(ω2, 0)+2),
[1 \\ 1, 2 \\ 2] has level φ(ω, φ(ω2, 0)+ω),
[1 \\ 1 [1 \\ 1 \\ 2] 2 \\ 2] has level φ(ω, φ(ω2, 0)2),
[1 \\ 1 [1 \\ 1 [1 \\ 1 \\ 2] 2 \\ 2] 2 \\ 2] has level φ(ω, φ(ω, φ(ω2, 0)2)),
[1 \\ 1 \ 2 \\ 2] has level φ(ω+1, φ(ω2, 0)+1),
[1 \\ 1 \ 1 \ ... \ 1 \ 2 \\ 2] (with n 1's between \\'s) has level φ(ω+n, φ(ω2, 0)+1).

When α is a limit ordinal and γ < α, φ(γ, φ(α, β)+1) has limit ordinal φ(α, β+1) as γ tends to α. The ordinal φ(ω+n, φ(ω2, 0)+1) therefore has limit ordinal φ(ω2, 1) as n tends to ω.

[1 \\ 1 \\ 3] has level φ(ω2, 1),
[1 \\ 1 \\ 4] has level φ(ω2, 2),
[1 \\ 1 \\ 1, 2] has level φ(ω2, ω),
[1 \\ 1 \\ 1 [1 \\ 1 \\ 2] 2] has level φ(ω2, φ(ω2, 0)),
[1 \\ 1 \\ 1 [1 \\ 1 \\ 1 [1 \\ 1 \\ 2] 2] 2] has level φ(ω2, φ(ω2, φ(ω2, 0))).

In fact, the development of [1 \\ 1 \\ X] separators mirrors the [1 \\ X] set except that φ(ω+n, α) is replaced by φ(ω2+n, α) in each of the associated ordinals. It follows that

[1 \\ 1 \\ 1 \ 2] has level φ(ω2+1, 0),
[1 \\ 1 \\ 1 \ 3] has level φ(ω2+1, 1),
[1 \\ 1 \\ 1 \ 4] has level φ(ω2+1, 2),
[1 \\ 1 \\ 1 \ 1 \ 2] has level φ(ω2+2, 0),
[1 \\ 1 \\ 1 \ 1 \ 3] has level φ(ω2+2, 1),
[1 \\ 1 \\ 1 \ 1 \ 1 \ 2] has level φ(ω2+3, 0),
[1 \\ 1 \\ 1 \ 1 \ 1 \ 1 \ 2] has level φ(ω2+4, 0),
[1 \\ 1 \\ 1 \ 1 \ ... \ 1 \ 2] (with n single backslashes) has level φ(ω2+n, 0),
[1 \\ 1 \\ 1 \ 1 \ ... \ 1 \ m] (with n single backslashes) has level φ(ω2+n, m-2).

Each additional double backslash adds ω to the left argument of the 2-ary φ Veblen function.

[1 \\ 1 \\ 1 \\ 2] has level φ(ω3, 0),
[1 \\ 1 \\ 1 \\ 3] has level φ(ω3, 1)   (limit ordinal of φ(ω2+n, φ(ω3, 0)+1) as n → ω),
[1 \\ 1 \\ 1 \\ m] has level φ(ω3, m-2),
[1 \\ 1 \\ 1 \\ 1 \ 2] has level φ(ω3+1, 0),
[1 \\ 1 \\ 1 \\ 1 \ 1 \ ... \ 1 \ m] (with n single backslashes) has level φ(ω3+n, m-2),
[1 \\ 1 \\ 1 \\ 1 \\ 2] has level φ(ω4, 0),
[1 \\ 1 \\ 1 \\ 1 \\ m] has level φ(ω4, m-2),
[1 \\ 1 \\ 1 \\ 1 \\ 1 \ 2] has level φ(ω4+1, 0),
[1 \\ 1 \\ 1 \\ 1 \\ 1 \ 1 \ ... \ 1 \ m] (with n single backslashes) has level φ(ω4+n, m-2),
[1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2] has level φ(ω5, 0).

In general, with n double backslashes,

[1 \\ 1 \\ ... \\ 1 \\ 2] has level φ(ωn, 0),
[1 \\ 1 \\ ... \\ 1 \\ k] has level φ(ωn, k-2),
[1 \\ 1 \\ ... \\ 1 \\ 1 \ 2] has level φ(ωn+1, 0),

[1 \\ 1 \\ ... \\ 1 \\ 1 \ 1 \ ... \ 1 \ k] (with m single backslashes) has level φ(ωn+m, k-2).

With c double backslashes and d single backslashes,
{a, b [1 \\ 1 \\ ... \\ 1 \\ 1 \ 1 \ 1 \ ... \ 1 \ 2] 2}
= {a ‹0 \\ 1 \\ ... \\ 1 \\ 1 \ 1 \ 1 \ ... \ 1 \ 2› b}
= {a ‹R_b› b},
where R_n = ‘b \ b \ ... \ b \\ b \ b \ ... \ b \\ ...... \\ b \ b \ ... \ b \\ b \ b \ ... \ b \ b ‹R_{n-1}› b’
(with b b’s and \’s followed by \\ (c times), followed by d-1 b’s and \’s, followed by ‘b ‹R_{n-1}› b’),
R_1 = ‘0’.

The first separator requiring the third in the sequence of hyperseparators (a treble backslash),
{a, b [1 \\\ 2] 2} = {a ‹0 \\\ 2› b}
= {a ‹R› b},
where   R = ‘b \ b \ ... \ b \\ b \ b \ ... \ b \\ ...... \\ b \ b \ ... \ b’
(with b^2 b’s and b-1 double backslashes).
This is a φ(ω^2, 0)-recursive function. The R string represents a 2-dimensional ‘hyperspace’ (or 2-hyperspace) of b^2 ‘hyperentries’ of b, separated by the backslashes.

I now denote the double backslash and the treble backslash by the symbols [2]\ and [3]\ respectively (the single backslash is a shorthand for [1]\). In general, the nth hyperseparator [n]\ denotes \\\...\ with n backslashes, and is analogous to the nth simple separator [n]; the [n]\ hyperseparator ranks above [m]\ whenever n > m. Hyperseparators are so special that they cannot appear in the ‘base layer’ of a main array (within curly brackets), as in {a, b \ 2}, {a, b [2]\ 2} and {a, b [n]\ 2} – this enables me to generate ‘normal’ separators (i.e. non-hyperseparators) with higher ordinal levels, and a more advanced notation that would be the case if hyperseparators had been allowed in the base layers of main arrays.

The array {a, b [1 \\\ 2] 2} can be rewritten as follows:
{a, b [1 [3]\ 2] 2} = {a ‹0 [3]\ 2› b}
= {a ‹R› b},
where   R = ‘b ‹2›\ b’
= ‘b ‹1›\ b [2]\ b ‹1›\ b [2]\ ...... [2]\ b ‹1›\ b’   (with b ‘b ‹1›\ b’ strings)
= ‘b \ b \ ... \ b [2]\ b \ b \ ... \ b [2]\ ...... [2]\ b \ b \ ... \ b’
(with b^2 b’s and b b’s within each [2]\).
Note that ‘b ‹0›\ b’ = ‘b’, just as ‘b ‹0› b’ = ‘b’.

The iterating string building function (R_n) is only utilised when the final separator prior to the next non-1 entry (after the initial 0) in the angle brackets is a single backslash (\ or [1]\).

When # denotes the rest of the array,
{a, b [1 [c]\ 1 [d]\ 1 [e]\ 1 [2]\ 1 \ 1 \ 1 \ k #] 2}
= {a ‹ 0 [c]\ 1 [d]\ 1 [e]\ 1 [2]\ 1 \ 1 \ 1 \ k # › b}
= {a ‹R_b› b},
where R_n = ‘b ‹c-1›\ b [c]\ b ‹d-1›\ b [d]\ b ‹e-1›\ b [e]\ b ‹1›\ b [2]\ b \ b \ b ‹R_{n-1}› b \ k-1 #’,
R_1 = ‘0’.
Note that c ≥ d ≥ e ≥ 2 as trailing 1’s are lopped off just as with simpler nested arrays (below ε_0 level).

When X is a string of characters within a normal separator (below \ level),
{a, b [1 [c]\ 1 [d]\ 1 [e]\ 1 [2]\ 1 \ 1 \ 1 \ 1 [X] k #] 2}

= {a ‹ 0 [c]\ 1 [d]\ 1 [e]\ 1 [2]\ 1 \ 1 \ 1 \ 1 [X] k # › b}
= {a ‹R› b},

where R = ‘b ‹c-1›\ b [c]\ b ‹d-1›\ b [d]\ b ‹e-1›\ b [e]\ b ‹1›\ b [2]\ b \ b \ b \ b ‹X’› b [X] k-1 #’ and X’ is identical to X except that the first entry has been reduced by 1.

When the last separator before the next non-1 entry (k) is a hyperseparator of [2]\ or higher,

{a, b [1 [c]\ 1 [d]\ 1 [e]\ 1 [2]\ k #] 2}
= {a ‹ 0 [c]\ 1 [d]\ 1 [e]\ 1 [2]\ k # › b}
= {a ‹R› b},

where R = ‘b ‹c-1›\ b [c]\ b ‹d-1›\ b [d]\ b ‹e-1›\ b [e]\ b ‹1›\ b [2]\ k-1 #’.

[1 [3]\ 2] has level φ(ω^2, 0),
[1 \ 2 [3]\ 2] has level $\varepsilon_{\varphi(\omega\text{^}2,\ 0)+1}$ = φ(ω^2, 0)^^ω,
[1 \ 1 \ 1 \ ... \ 1 \ 2 [3]\ 2] (with n 1’s) has level φ(n, φ(ω^2, 0)+1),
[1 [2]\ 2 [3]\ 2] has level φ(ω, φ(ω^2, 0)+1),
[1 [2]\ 1 [2]\ ... [2]\ 1 [2]\ 2 [3]\ 2] (with n 1’s) has level φ(ωn, φ(ω^2, 0)+1),
[1 [3]\ 3] has level φ(ω^2, 1) (limit ordinal of φ(ωn, φ(ω^2, 0)+1) as n → ω),
[1 [3]\ 1, 2] has level φ(ω^2, ω),
[1 [3]\ 1 \ 2] has level φ(ω^2+1, 0),
[1 [3]\ 1 \ 1 \ 2] has level φ(ω^2+2, 0),
[1 [3]\ 1 [2]\ 2] has level φ(ω^2+ω, 0),
[1 [3]\ 1 [2]\ 1 [2]\ 2] has level φ(ω^2+ω2, 0),
[1 [3]\ 1 [3]\ 2] has level φ((ω^2)2, 0),
[1 [3]\ 1 [3]\ 1 [3]\ 2] has level φ((ω^2)3, 0),
[1 [4]\ 2] has level φ(ω^3, 0),
[1 [5]\ 2] has level φ(ω^4, 0),
[1 [n]\ 2] has level φ(ω^(n-1), 0).

Now the number in the hyperseparator can itself be turned into an array:

[1 [1, 2]\ 2] has level φ(ω^ω, 0),
[1 [n] 2 [1, 2]\ 2] has level φ(ω^(n-1), φ(ω^ω, 0)+1),
[1 [1, 2]\ 3] has level φ(ω^ω, 1) (limit ordinal of φ(ω^n, φ(ω^ω, 0)+1) as n → ω),
[1 [1, 2]\ 1, 2] has level φ(ω^ω, ω),
[1 [1, 2]\ 1 \ 2] has level φ(ω^ω+1, 0),
[1 [1, 2]\ 1 [2]\ 2] has level φ(ω^ω+ω, 0),
[1 [1, 2]\ 1 [3]\ 2] has level φ(ω^ω+ω^2, 0),
[1 [1, 2]\ 1 [n]\ 2] has level φ(ω^ω+ω^(n-1), 0),
[1 [1, 2]\ 1 [1, 2]\ 2] has level φ((ω^ω)2, 0),
[1 [2, 2]\ 2] has level φ((ω^ω)ω, 0) = φ(ω^(ω+1), 0),
[1 [3, 2]\ 2] has level φ(ω^(ω+2), 0),
[1 [1, 3]\ 2] has level φ(ω^(ω2), 0),
[1 [1, 4]\ 2] has level φ(ω^(ω3), 0),
[1 [1, 1, 2]\ 2] has level φ(ω^ω^2, 0),
[1 [1, 1, 3]\ 2] has level φ(ω^((ω^2)2), 0),
[1 [1, 1, 1, 2]\ 2] has level φ(ω^ω^3, 0),
[1 [1, 1, 1, 1, 2]\ 2] has level φ(ω^ω^4, 0),
[1 [1 [2] 2]\ 2] has level φ(ω^ω^ω, 0),
[1 [1 [3] 2]\ 2] has level φ(ω^ω^ω^2, 0),
[1 [1 [1, 2] 2]\ 2] has level φ(ω^ω^ω^ω, 0),

[1 [1 [1 [2] 2] 2]\ 2] has level $\varphi(\omega\string^\omega\string^\omega\string^\omega\string^\omega, 0)$,
[1 [1 [1 [1, 2] 2] 2]\ 2] has level $\varphi(\omega\string^\omega\string^\omega\string^\omega\string^\omega\string^\omega, 0)$.

If [X] has level α then [1 [X]\ 2] has level $\varphi(\omega\string^\alpha, 0)$. I can now nest hyperseparator arrays within hyperseparator arrays – the foundations of a Hyper-Nested Array Notation. This is where the mathematics gets so exciting!

[1 [1 \ 2]\ 2] has level $\varphi(\varepsilon_0, 0)$,
[1 [1 \ 2]\ 3] has level $\varphi(\varepsilon_0, 1)$ (limit ordinal of $\varphi(\alpha, \varphi(\varepsilon_0, 0)+1)$ as $\alpha \rightarrow \varepsilon_0$),
[1 [1 \ 2]\ 1, 2] has level $\varphi(\varepsilon_0, \omega)$,
[1 [1 \ 2]\ 1 \ 2] has level $\varphi(\varepsilon_0+1, 0)$,
[1 [1 \ 2]\ 1 [1 \ 2]\ 2] has level $\varphi(\varepsilon_0 2, 0)$,
[1 [2 \ 2]\ 2] has level $\varphi(\varepsilon_0\omega, 0)$,
[1 [1, 2 \ 2]\ 2] has level $\varphi(\varepsilon_0\omega\string^\omega, 0)$,
[1 [1 [1 \ 2] 2 \ 2]\ 2] has level $\varphi(\varepsilon_0\string^2, 0)$,
[1 [1 \ 3]\ 2] has level $\varphi(\varepsilon_1, 0)$,
[1 [1 \ 1, 2]\ 2] has level $\varphi(\varepsilon_\omega, 0)$,
[1 [1 \ 1 [1 \ 2] 2]\ 2] has level $\varphi(\varepsilon(\varepsilon_0), 0)$,
[1 [1 \ 1 \ 2]\ 2] has level $\varphi(\zeta_0, 0) = \varphi(\varphi(2, 0), 0)$,
[1 [1 \ 1 \ 1 \ 2]\ 2] has level $\varphi(\varphi(3, 0), 0)$,
[1 [1 [2]\ 2]\ 2] has level $\varphi(\varphi(\omega, 0), 0)$,
[1 [1 [3]\ 2]\ 2] has level $\varphi(\varphi(\omega\string^2, 0), 0)$,
[1 [1 [1, 2]\ 2]\ 2] has level $\varphi(\varphi(\omega\string^\omega, 0), 0)$,
[1 [1 [1 [2] 2]\ 2]\ 2] has level $\varphi(\varphi(\omega\string^\omega\string^\omega, 0), 0)$,
[1 [1 [1 \ 2]\ 2]\ 2] has level $\varphi(\varphi(\varepsilon_0, 0), 0)$,
[1 [1 [1 [2]\ 2]\ 2]\ 2] has level $\varphi(\varphi(\varphi(\omega, 0), 0), 0)$,
[1 [1 [1 [1 \ 2]\ 2]\ 2]\ 2] has level $\varphi(\varphi(\varphi(\varepsilon_0, 0), 0), 0)$,
[1 [1 [1 [1 [1 \ 2]\ 2]\ 2]\ 2]\ 2] has level $\varphi(\varphi(\varphi(\varphi(\varepsilon_0, 0), 0), 0), 0)$,
[1 [1 [1 [1 [1 [1 \ 2]\ 2]\ 2]\ 2]\ 2]\ 2] has level $\varphi(\varphi(\varphi(\varphi(\varphi(\varepsilon_0, 0), 0), 0), 0), 0)$.
The sequence of separators starting with the last three has limit ordinal $\Gamma_0 = \varphi(1, 0, 0)$.

The Main Rules and Angle Bracket Rules apply similarly to hyperseparators and hyper-angle bracket arrays (each signified by a single backslash after the closed bracket), just as they do to normal separators and angle bracket arrays (i.e. with no backslash after the closed bracket). There are three additional Angle Bracket (B) Rules when the first hyperentry (space before first hyperseparator) contains a single 0:-

Rule B1 (only 1 entry of either 0 or 1 within hyper-angle bracket array):
‘a ‹0›\ b’ = ‘a’,
‘a ‹1›\ b’ = ‘a \ a \ ... \ a’ (with b a’s).

Rule B2 (first hyperentry is 0, next non-1 entry (k) is the first entry in its hyperentry but is not contained in the first hyperentry in its 1-hyperspace):
‘a ‹ 0 [A_1]\ 1 [A_2]\ ... 1 [A_p]\ 1 \ k # ›(\) b’ = ‘a ‹R_b›(\) b’,
where R_n = ‘b ‹A_1’›\ b [A_1]\ b ‹A_2’›\ b [A_2]\ ... b ‹A_p’›\ b [A_p]\ b ‹R_{n-1}› b \ k-1 #’ ($n > 1$),
R_1 = ‘0’.
($p \geq 0$. Final separator before k is \ or [1]\.)
If p = 0, then

‘a ‹ 0 \ k # ›(\) b’ = ‘a ‹b ‹b ‹ ... ‹b ‹b \ k-1 #› b \ k-1 #› ... › b \ k-1 #› b \ k-1 #›(\) b’
(with b-1 angle brackets).

Rule B3 (Rules B1-2 do not apply, first hyperentry is 0):

‘a ‹ 0 [A_1]\ 1 [A_2]\ ... 1 [A_p]\ 1 [B_1] 1 [B_2] ... 1 [B_q] k # ›(\) b’
= ‘a ‹ b ‹A_1’›\ b [A_1]\ b ‹A_2’›\ b [A_2]\ ... b ‹A_p’›\ b [A_p]\
b ‹B_1’› b [B_1] b ‹B_2’› b [B_2] ... b ‹B_q’› b [B_q] k-1 # ›(\) b’.

($p \geq 1$, $q \geq 0$. ($p = 0$ would be identical to Rule A4.) Final separator before k is not \ or [1]\.)

Notes:

1. A_1, A_2, ... , A_p are strings of characters within hyperseparators.
2. B_1, B_2, ... , B_q are strings of characters within normal separators.
3. A_1’, A_2’, ... , A_p’, B_1’, B_2’, ... , B_q’ are strings of characters within angle brackets that are identical to the strings A_1, A_2, ... , A_p, B_1, B_2, ... , B_q respectively except that the first entries of each have been reduced by 1. If A_i (for some $1 \leq i \leq p$) begins with 1, A_i’ begins with 0.
4. R_n is an iterating string building function which creates and nests the same string of characters around itself n-1 times before being replaced by the string ‘0’.
5. # is a string of characters representing the remainder of the array (if it exists).
6. The comma is used as shorthand for the [1] separator.
7. The isolated backslash \ is used as shorthand for the [1]\ hyperseparator.
8. A backslash in brackets (\) immediately after the closed outer angle bracket denotes that when the left-hand side (LHS) of the equation contains a backslash after the closed outer angle bracket then so does the right-hand side (RHS). On the other hand, if the LHS does not contain that backslash then the RHS does not either.

For example, (\) in Rule B3 (with p = q = 1) means that:

‘a ‹ 0 [A_1]\ 1 [B_1] k # › b’ = ‘a ‹ b ‹A_1’›\ b [A_1]\ b ‹B_1’› b [B_1] k-1 # › b’

and ‘a ‹ 0 [A_1]\ 1 [B_1] k # ›\ b’ = ‘a ‹ b ‹A_1’›\ b [A_1]\ b ‹B_1’› b [B_1] k-1 # ›\ b’.

Modifications to the regular Angle Bracket (A) Rules:

The backslash in brackets (\) can be appended to the closed outer angle brackets in Rules A2-5 (including closed square brackets in Rule A5) similarly as above. For example, in Rule A4 (with k = 2),

‘a ‹0 [A_1] 1 [A_2] n #› b’ = ‘a ‹ b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] n-1 # › b’

and ‘a ‹0 [A_1] 1 [A_2] n #›\ b’ = ‘a ‹ b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] n-1 # ›\ b’.

Rule A2 is modified as follows:

‘a ‹# [A] 1›(\) b’ = ‘a ‹#›(\) b’,
‘a ‹# [A]\ 1›(\) b’ = ‘a ‹#›(\) b’,
‘a ‹# [A] 1 [B]\ #*›(\) b’ = ‘a ‹# [B]\ #*›(\) b’ (regardless of the levels of [A] and [B]).

When level of [A] is less than level of [B],

‘a ‹# [A] 1 [B] #*›(\) b’ = ‘a ‹# [B] #*›(\) b’,
‘a ‹# [A]\ 1 [B]\ #*›(\) b’ = ‘a ‹# [B]\ #*›(\) b’.

This is the first stage of my Hyper-Nested Array Notation. The limit ordinal of this notation is Γ_0, the Feferman-Schütte ordinal.

A separator is generally of the form

[A_1 [H_1]\ A_2 [H_2]\ ... [H_{k-1}]\ A_k],

where A_i = ‘$n_{i,1}$ [$S_{i,1}$] $n_{i,2}$ [$S_{i,2}$] ... [S_{i,p_i-1}] n_{i,p_i}’
and k ≥ 1, p_i ≥ 1, each of [H_i]\ is an hyperseparator, each of [$S_{i,j}$] is a normal separator and each of $n_{i,j}$ represents an entry (number). A hyperseparator is as above but of the form

[A_1 [H_1]\ A_2 [H_2]\ ... [H_{k-1}]\ A_k]\

with a backslash added to the right of the outermost closed square bracket.

In Rules M2 and A2, the (ordinal) levels of two normal separators [A] and [B] (or hyperseparators [A]\ and [B]\) are determined by the highest ranking hyperseparator within their ‘base layers’, then the numbers of them when they are identical. When the numbers are equal, this is repeated for the subarrays of [A] and [B] to the right of the rightmost highest ranking hyperseparator. When the highest ranking hyperseparators and their numbers within the subarrays are identical, this is repeated again for the subarrays within the subarrays, until no more hyperseparators remain, in which case the levels of [A] and [B] are determined by the highest ranking normal separator within their ‘base layers’, then their numbers when identical. When [A] and [B] are still tied on that measure, this is repeated for the subarrays of [A] and [B] to the right of the last highest separator, and, if still equal, repeated again for the subarrays within the subarrays, until no more separators remain (i.e. we are left with single entries). If the final entries of [A] and [B] are the same, they are deleted along with the last separators of [A] and [B], and the entire process is repeated for the truncated [A] and [B], until each of these consists of a single entry. When we obtain a lower level or number for [A] than for [B] on some measure at a particular point, then the original [A] ranks lower than the original [B] and the ‘[A] 1’ string is deleted. (The levels of separators or hyperseparators within the ‘base layers’ of [A] and [B] are first determined by the highest ranking hyperseparator (normal separator in the absence of hyperseparators) within their ‘base layers’ (in the second lowest layers of [A] and [B]), and so on.)

The most complicated rule is Rule B2. For example,

{3, 2 [1 [3]\ 1 [2]\ 1 \ 1 \ 1 \ 7] 2} = {3 ‹ 0 [3]\ 1 [2]\ 1 \ 1 \ 1 \ 7 › 2}
= {3 ‹R› 2},

where R = ‘2 ‹2›\ 2 [3]\ 2 ‹1›\ 2 [2]\ 2 \ 2 \ 2 \ 6’
= ‘2 \ 2 \\ 2 \ 2 \\\ 2 \ 2 \\ 2 \ 2 \ 2 \ 6’.

{3, 3 [1 [3]\ 1 [2]\ 1 \ 1 \ 1 \ 7] 2} = {3 ‹ 0 [3]\ 1 [2]\ 1 \ 1 \ 1 \ 7 › 3}
= {3 ‹R› 3},

where R = ‘3 ‹2›\ 3 [3]\ 3 ‹1›\ 3 [2]\ 3\3\ 3‹S›3 \6’
= ‘3\3\3 \\ 3\3\3 \\ 3\3\3 \\\ 3\3\3 \\ 3\3\ 3‹S›3 \6’,
S = ‘3\3\3 \\ 3\3\3 \\ 3\3\3 \\\ 3\3\3 \\ 3\3\3\6’.

{3, 4 [1 [3]\ 1 [2]\ 1 \ 1 \ 1 \ 7] 2} = {3 ‹ 0 [3]\ 1 [2]\ 1 \ 1 \ 1 \ 7 › 4}
= {3 ‹R› 4},

where R = ‘4 ‹2›\ 4 [3]\ 4 ‹1›\ 4 [2]\ 4\4\ 4‹S›4 \6’
= ‘4\4\4\4 \\ 4\4\4\4 \\ 4\4\4\4 \\ 4\4\4\4 \\\ 4\4\4\4 \\ 4\4\ 4‹S›4 \6’,
S = ‘4\4\4\4 \\ 4\4\4\4 \\ 4\4\4\4 \\ 4\4\4\4 \\\ 4\4\4\4 \\ 4\4\ 4‹T›4 \6’,
T = ‘4\4\4\4 \\ 4\4\4\4 \\ 4\4\4\4 \\ 4\4\4\4 \\\ 4\4\4\4 \\ 4\4\4\6’.

But I can extend my Hyper-Nested Array Notation even further! Using the forward slash (for want of a better notation), I can define

{a, b [1 / 2] 2} = {a ‹0 / 2› b}
= {a ‹b ‹b ‹ ... ‹b ‹b›\ b›\ ... ›\ b›\ b› b}
(with b b’s from centre to right and b-2 backslashes).

This is equivalent to

{a, b [1 [1 [... [1 [1, 2]\ 2]\ ...]\ 2]\ 2] 2}

(with b-1 square brackets and b-2 backslashes, for b ≥ 3)

and is a Γ_0-recursive function.

For example,

{3, 2 [1 / 2] 2} = {3 ‹0 / 2› 2}
= {3 ‹2› 2}
= {3, 3 [2] 3, 3},
{3, 3 [1 / 2] 2} = {3 ‹0 / 2› 3}
= {3 ‹3 ‹3›\ 3› 3}
= {3 ‹ 3\3\3 \\ 3\3\3 \\ 3\3\3 [3]\ 3\3\3 \\ 3\3\3 \\ 3\3\3 [3]\ 3\3\3 \\ 3\3\3 \\ 3\3\3 › 3},
{3, 4 [1 / 2] 2} = {3 ‹0 / 2› 4}
= {3 ‹4 ‹4 ‹4›\ 4›\ 4› 4},
{3, 5 [1 / 2] 2} = {3 ‹0 / 2› 5}
= {3 ‹5 ‹5 ‹5 ‹5›\ 5›\ 5›\ 5› 5}.

The separator [1 / 2] has level Γ_0. But, already, this has made me feel I'm halfway there to infinity!

I have devised this monster notation to try to explain the truly astonishing growth rate of Friedman's TREE sequence (finite form of Kruskal's Tree Theorem). There's no exaggeration whatsoever in the fact that its growth rate exceeds that of

f(n) = {3, n [1 / 2] 2}

which is at Γ_0 level. However, the ordinal measuring the strength of Kruskal's theorem is the small Veblen ordinal, which is also the ordinal type of a certain ordering of rooted trees (Jervell 2005).

This notation is truly amazing! The introduction of the single backslash got me to ε_0, the double backslash got me to $\varphi(\omega, 0)$, the single backslash within a backslash array got me to $\varphi(\varepsilon_0, 0)$, and now, the nested backslash array has blasted me right up to Γ_0. What next? Nested forward slash arrays, nested third level slash arrays, even the slash level itself being represented by nested slash level arrays! I aim to reach the small Veblen ordinal and beyond, in Part II of Beyond Bird's Nested Arrays.

The number

N = {3, 3 [1 / 2] 2}

is big enough, but can you imagine for a moment how huge the number

{3, 3, 2 [1 / 2] 2} = {3, N [1 / 2] 2}

is, bearing in mind that when the array is written out in full with entries of 3's and separator arrays we have only to reduce the number in the third entry by one for the second entry to contain a number so large that it represents a copy of the *entire array* with the only change being that its second entry has been reduced by one (Rule M7)?

If there are more than {3, N [1 / 2] 2} subatomic particles in the entire multiverse or hyperuniverse then there must truly be an infinite number of them!

Author: Chris Bird (Gloucestershire, England, UK)
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E-mail: m.bird44 at btinternet.com (not clickable to thwart spambots!)