Floretions 2009, DRAFT

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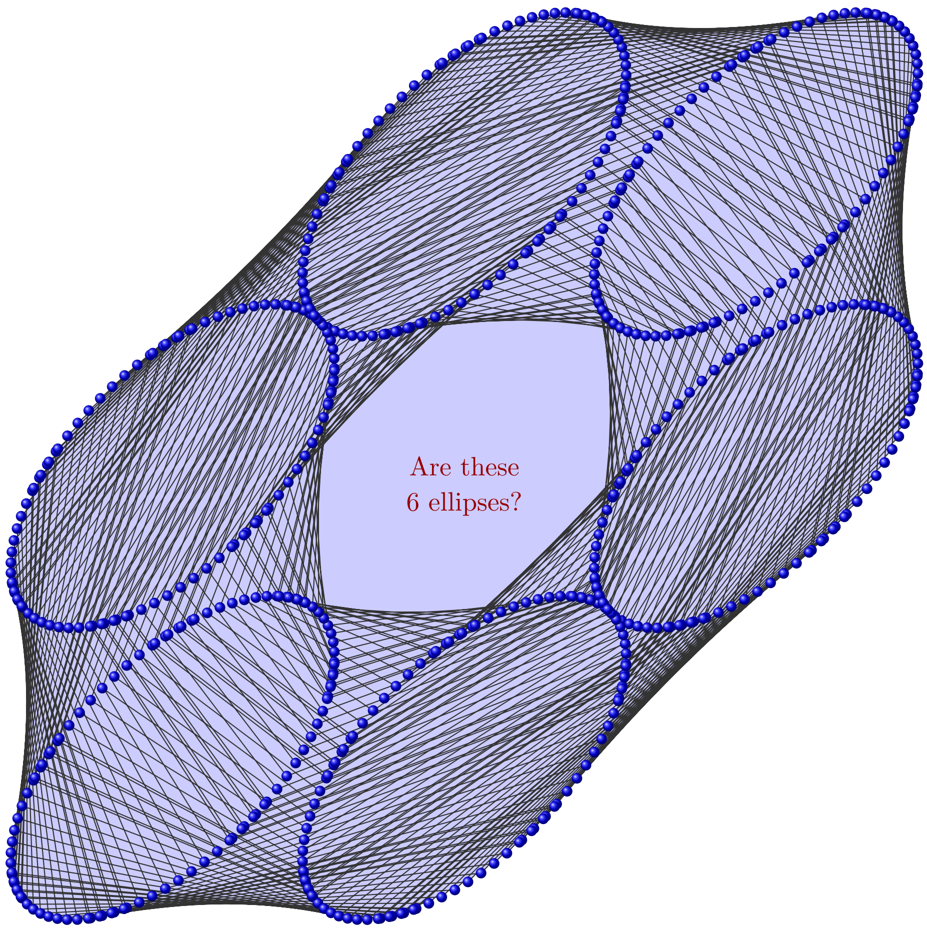


1 Abstract

The goal of this paper is to establish and explore the first four of the following six points which deal with floretions:...

- Conditions under which a given floretion leads to a linear recurrence relation of 4th order or less, in particular the conditions for the generation of a sequence of integers for static types.

- Static and dynamic identities. With the help of these identities, we will see that it is nearly possible to automate the finding of simple relationships between Fibonacci, Lucas, and Pell numbers.

- Using Floret's Cube to make "intelligent" guesses on how to choose base coefficients.

- Geometrical aspects. Examples: 1. How are the points and line segments in the above picture (for example) generated and do the discrete, ellipse-like objects actually represent real conic sections?

- Necklace aspects and prime numbers.

- Types of convergence and Transforms

Floretions can be utilized to generate several types of sequences. Here is an overview- where the last column refers to whether there are established connections between a given type and Triangular/Fibonacci/Pell numbers:

Type Name	Tri / Fib / Pell
4th Order Linear Recurrence or less (static type)	yes
16th Order or less (dynamic type)	yes
Necklace	yes
Geometric (sum type)	unknown
Transforms (force type)	yes

Several tools are available to readers of this article:

- Online Floretion Multiplier

- Floretion Symbolic Multiplier, a more powerful version of the Online Multiplier. If you spot `>>> some command` anywhere in this paper, that is a reference to IDLE's Python command line. The online version of this program currently has symbolic multiplication (i.e. variables) disabled as it uses up too many recourses on the server side.

- Structure of the Floretion Group (R. Mathar)

- Sequences Related to Floretions (R. Munafo)

- Import Floretions script for Blender3D which allows floretion curves to be generated two and three dimensions. Curves which lead to ellipses as in the top graphic will be discussed in the final chapter.

- Prof. E. Clark's "Floretion" Maple Script, where he demonstrates that the 16 floretion (positive) basis vectors form a basis for the space of 4×4 real matrices.

2 Introduction

2.1 Notation and Definitions

The 16 (positive) **floretion basis elements** can be written in several notations, depending on available fonts and suitability. Note that the unit vector $\overleftrightarrow{ee}$ is also written in bold as **1**.

Left/Right Notation	$\overleftarrow{i}$	$\overleftarrow{j}$	$\overleftarrow{k}$	$\overrightarrow{i}$	$\overrightarrow{j}$	$\overrightarrow{k}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{jj}$
	$\overleftrightarrow{kk}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{ik}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{jk}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{kj}$	$\overleftrightarrow{ee}$
Left/Right Notation Ascii-Version	'i	'j	'k	i'	j'	k'	$'ii'$	$'jj'$
	$'kk'$	$'ij'$	$'ik'$	$'ji'$	$'jk'$	$'ki'$	$'kj'$	ee
Matrix/Ascii/Python	IE	JE	KE	EI	EJ	EK	II	JJ
	KK	IJ	IK	JI	JK	KI	KJ	EE
Factor Space Notation	$[(i,1)]$	$[(j,1)]$	$[(k,1)]$	$[(1,i)]$	$[(1,j)]$	*etc.*		

Accounting for elements with a negative sign, there are 32 group elements in all. We begin by reviewing basic multiplication in the "left/right" notation.

$$\overleftrightarrow{ee} \cdot \overleftarrow{i} = \overleftarrow{i} \cdot \overleftrightarrow{ee} = \overleftarrow{i}$$

Note: $\overleftrightarrow{ee}$ is the unit vector.

$$\overleftarrow{i} \cdot \overleftarrow{i} = -\overleftrightarrow{ee}$$
$$\overleftarrow{j} \cdot \overleftarrow{j} = -\overleftrightarrow{ee}$$
$$\overleftarrow{k} \cdot \overleftarrow{k} = -\overleftrightarrow{ee}$$

$$\overleftarrow{i} \cdot \overleftarrow{j} = \overleftarrow{k}$$
$$\overleftarrow{j} \cdot \overleftarrow{i} = -\overleftarrow{k}$$

$$\overleftarrow{j} \cdot \overleftarrow{k} = \overleftarrow{i}$$
$$\overleftarrow{k} \cdot \overleftarrow{j} = -\overleftarrow{i}$$

$$\overleftarrow{k} \cdot \overleftarrow{i} = \overleftarrow{j}$$

$\overleftarrow{i} \cdot \overleftarrow{k} = -\overleftarrow{j}$

The set $\{\pm\overleftarrow{i}, \pm\overleftarrow{j}, \pm\overleftarrow{k}, \pm\overleftrightarrow{ee}\}$ is isomorph to the quaternion group $\mathcal{Q}$. The set $\{\pm\overrightarrow{i}, \pm\overrightarrow{j}, \pm\overrightarrow{k}, \pm\overleftrightarrow{ee}\}$ is also isomorph to the quaternion group. Just as before, we have $\overrightarrow{i} \cdot \overrightarrow{j} = \overrightarrow{k}$, etc. The remaining basis vectors are defined as:

$\overleftarrow{i} \cdot \overrightarrow{i} = \overrightarrow{i} \cdot \overleftarrow{i} = \overleftrightarrow{ii}$
$\overleftarrow{i} \cdot \overrightarrow{j} = \overrightarrow{j} \cdot \overleftarrow{i} = \overleftrightarrow{ij}$
$\overleftarrow{i} \cdot \overrightarrow{k} = \overrightarrow{k} \cdot \overleftarrow{i} = \overleftrightarrow{ik}$
etc.

In other words, if an element is chosen from the set $\{\pm\overleftarrow{i}, \pm\overleftarrow{j}, \pm\overleftarrow{k}, \pm\overleftrightarrow{ee}\}$ and a second element from the set $\{\pm\overrightarrow{i}, \pm\overrightarrow{j}, \pm\overrightarrow{k}, \pm\overleftrightarrow{ee}\}$, these two elements commute. Other products follow from the associative law and the above, for example:

$$\begin{aligned}
\overleftrightarrow{ii} \cdot \overleftrightarrow{ij} &= \overleftarrow{i} \cdot \overrightarrow{i} \cdot \overleftarrow{i} \cdot \overrightarrow{j} \\
&= \overleftarrow{i} \cdot (\overrightarrow{i} \cdot \overleftarrow{i}) \cdot \overrightarrow{j} \\
&= \overleftarrow{i} \cdot (\overleftarrow{i} \cdot \overrightarrow{i}) \cdot \overrightarrow{j} \\
&= (\overleftarrow{i} \cdot \overleftarrow{i}) \cdot (\overrightarrow{i} \cdot \overrightarrow{j}) \\
&= -\overleftrightarrow{ee} \cdot \overrightarrow{k} \\
&= -\overrightarrow{k}
\end{aligned}$$

It turns out that the above set of elements forms a group which is isomorph to the factor space $\mathcal{F} = \mathcal{Q} \times \mathcal{Q}/\{(1,1),(-1,-1)\}$ (see Acknowledgements section). Just as we can switch back and forth viewing the elements 1 and $\overleftarrow{i}$ as either members of the group $\{\pm 1, \pm\overleftarrow{i}\}$ or as a basis for the set $\mathbb{C}$ of complex numbers, so too can we do this with floretion basis vectors. Luckily for our purposes in this paper, the question of whether we are, in some particular case at hand, viewing quaternions/floretions as members of a finite group or as basis vectors making up an algebra is often either irrelevant or immediately clear upon inspection. Therefore, we shall not be using two sets of different notation for each case.

In this paper, the 16 elements without a negative sign are referred to as the *floretion basis (or base) elements*. Other terms which might also be used are *basis vectors* or *floretion group elements* (this includes elements with a negative sign).

We shall be particularly interested in multiplying elements of the form

$$
\begin{aligned}
X &= A\overleftarrow{i} + B\overleftarrow{j} + C\overleftarrow{k} + D\overrightarrow{i} + E\overrightarrow{j} + F\overrightarrow{k} + G\overleftrightarrow{ii} + H\overleftrightarrow{jj} + I\overleftrightarrow{kk} \\
&\quad + J\overleftrightarrow{ij} + K\overleftrightarrow{ik} + L\overleftrightarrow{ji} + M\overleftrightarrow{jk} + N\overleftrightarrow{ki} + O\overleftrightarrow{kj} + P\stackrel{\leftleftarrows}{ee} \\
&= \mathrm{ibase}(X)\overleftarrow{i} + \mathrm{jbase}(X)\overleftarrow{j} + \mathrm{kbase}(X)\overleftarrow{k} + \mathrm{basei}(X)\overrightarrow{i} + \mathrm{basej}(X)\overrightarrow{j} \\
&\quad + \mathrm{basek}(X)\overrightarrow{k} + \mathrm{ibasei}(X)\overleftrightarrow{ii} + \mathrm{jbasej}(X)\overleftrightarrow{jj} + \mathrm{kbasek}(X)\overleftrightarrow{kk} \\
&\quad + \mathrm{ibasej}(X)\overleftrightarrow{ij} + \mathrm{ibasek}(X)\overleftrightarrow{ik} + \mathrm{jbasei}(X)\overleftrightarrow{ji} + \mathrm{jbasek}(X)\overleftrightarrow{jk} \\
&\quad + \mathrm{kbasei}(X)\overleftrightarrow{ki} + \mathrm{kbasej}(X)\overleftrightarrow{kj} + \mathrm{tes}(X)\stackrel{\leftleftarrows}{ee}
\end{aligned}
$$

Any such element will be called a *floretion*. It can be shown that, written in matrix form (see Acknowledgements section), the set of base floretions $\overleftarrow{i}, \overleftarrow{j}$ form a basis for the linear space of 4×4 real matrices. For X, X^2, X^3 to lead to a sequence of integers in the context to be described, the condition: "$4A, 4B, \cdots, 4P$ are integers" is necessary but not sufficient. Thus, the only fractional parts allowed for all base coefficients are $\{0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}\}$. **Since our primary concern in the beginning is to generate integer sequences, the sections Introduction and Floret's Cube implicitely assume this restricted property in all statements which begin "Let X be a floretion".**

The set of all floretions which induce or "generate" integer sequences (under so-called static conditions- this will become clear shortly) is denoted by $\mathcal{Z}^\infty$. In particular, $X \in \mathcal{Z}^\infty \Leftrightarrow \forall n \in \mathbb{N} : \langle A_n \rangle, \langle B_n \rangle, \ldots \in \{0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}\}$ where $X^n = A_n\overleftarrow{i} + B_n\overleftarrow{j} + C_n\overleftarrow{k} + D_n\overrightarrow{i} + E_n\overrightarrow{j} + F_n\overrightarrow{k} + G_n\overleftrightarrow{ii} + H_n\overleftrightarrow{jj} + I_n\overleftrightarrow{kk} + J_n\overleftrightarrow{ij} + K_n\overleftrightarrow{ik} + L_n\overleftrightarrow{ji} + M_n\overleftrightarrow{jk} + N_n\overleftrightarrow{ki} + O_n\overleftrightarrow{kj} + P_n\stackrel{\leftleftarrows}{ee}$.

Note that $\mathcal{Z}^\infty$ is not "closed" in the sense that $X \cdot Y \in \mathcal{Z}^\infty$ does not necessarily follow from $X, Y \in \mathcal{Z}^\infty$. Interestingly, by Corollary 2.27, if $X \cdot Y \in \mathcal{Z}^\infty$ we can show $Y \cdot X \in \mathcal{Z}^\infty$ (note: one particular case of this corollary still relies on a conjecture). Nevertheless, the reader is invited to try and find two floretions, each in $\mathcal{Z}^\infty$, for which the product is not in $\mathcal{Z}^\infty$. This is a nice warm-up exercise and, even with the Floretion Multiplier, it may take a bit of guesswork to stumble onto an example. Harder yet, try to find an example where $\forall n : X^n \neq 0$, $X^2 \in \mathcal{Z}^\infty$ but $X \notin \mathcal{Z}^\infty$.

Definition 2.1 *Any floretion may be defined as a PYTHON dictionary as follows:*

```
X =
   {'ie': 'A', 'je': 'B', 'ke': 'C',
    'ei': 'D', 'ej': 'E', 'ek': 'F',
    'ii': 'G', 'jj': 'H', 'kk': 'I',
    'ij': 'J', 'ik': 'K', 'ji': 'L',
```

```
 'jk': 'M', 'ki': 'N', 'kj': 'O', 'ee': 'P'

 }
```

where A, B, C,... are the real coefficients of the 16 basis vectors ie, je, ke, etc.

The following definitions are fundamental to what follows:

```
ves(X) = A+B+C+D+E+F+G+H+I+J+K+L+M+N+O+P
jesleft(X) = A+B+C
jesright(X) = D+E+F
jes(X) = A+B+C+D+E+F
les(X) = G+H+I+J+K+L+M+N+O
tes(X) = P

and

ibase(X) = A, jbase(X) = B, kbase(X) = C
basei(X) = D, basej(X) = E, basek(X) = F
ibasei(X) = G, jbasej(X) = H, kbasek(X) = I
ibasej(X) = J, ibasek(X) = K
jbasei(X) = L, jbasek(X) = M
kbasei(X) = N, kbasej(X) = O
```

along with the projection operators VES (= id), JES, LES, TES, JESRIGHT, JESLEFT, etc.:

```
VES(X) = X
JES(X) =
    {'ie': 'A', 'je': 'B', 'ke': 'C',
     'ei': 'D', 'ej': 'E', 'ek': 'F',
     'ii': '', 'jj': '', 'kk': '',
     'ij': '', 'ik': '', 'ji': '',
     'jk': '', 'ki': '', 'kj': '', 'ee': ''

     }
LES(X) =
    {'ie': '', 'je': '', 'ke': '',
     'ei': '', 'ej': '', 'ek': '',
     'ii': 'G', 'jj': 'H', 'kk': 'I',
     'ij': 'J', 'ik': 'K', 'ji': 'L',
     'jk': 'M', 'ki': 'N', 'kj': 'O', 'ee': ''

     }
FAM(X) =
```

```
    {'ie': '', 'je': '', 'ke': '',
     'ei': '', 'ej': '', 'ek': '',
     'ii': 'G', 'jj': 'H', 'kk': 'I',
     'ij': '', 'ik': '', 'ji': '',
     'jk': '', 'ki': '', 'kj': '', 'ee': ''

     }
TES(X) =
    {'ie': '', 'je': '', 'ke': '',
     'ei': '', 'ej': '', 'ek': '',
     'ii': '', 'jj': '', 'kk': '',
     'ij': '', 'ik': '', 'ji': '',
     'jk': '', 'ki': '', 'kj': '', 'ee': 'P'

     } = VES(X) - JES(X) - LES(X)

FAMTES(X) = FAM(X) + TES(X)
```

The function "ves" adds all the coefficients:

```
ves(VES(X)) = ves(X)
ves(JES(X)) = jes(X)
```

etc. We define vesseq as the sequence:

```
vesseq(X) = (ves(X), ves(X^2), ves(X^3), ves(X^4), ...)
jesseq(X) = (jes(X), jes(X^2), jes(X^3), jes(X^4), ...)
lesseq(X) = (les(X), les(X^2), les(X^3), les(X^4), ...)
etc.
```

For future reference, note that

```
vesseq(X)(n) = ves(X^n)
```

From the relation

```
 ves(X) = jes(X) + les(X) + tes(X)
```

we immediately have an identity which relates 4 sequences of real numbers, namely:

Proposition 2.2 Static Identities

```
vesseq(X) = jesseq(X) + lesseq(X) + tesseq(X)
```

Many similar static identities are used. For example, the reader may easily check that

```
jesseq(X) = ibaseseq(X) + jbaseseq(X) + kbaseseq(X) +
baseiseq(X) + basejseq(X) + basekseq(X)
```

In fact, in many cases these 4 sequences are sequences of integers- though often all terms of a given sequence must first be multiplied by a factor of 2 or 4. A trivial case would be:

```
vesseq(X) = (1,1,1,1,..)
jesseq(X) = (0.5,0.5,0.5,0.5,)
lesseq(X) = (0.25,0.25,0.25,0.25,)
tesseq(X) = (0.25,0.25,0.25,0.25,)
```

In this case, the command seqFinder(X) will automatically return integer sequences by multiplying `jesseq` *by 2 and* `lesseq/tesseq` *by 4:*

```
vesseq(X) = (1,1,1,1,..)
2jesseq(X) = (1,1,1,1,..)
4lesseq(X) = (1,1,1,1,..)
4tesseq(X) = (1,1,1,1,..)
```

Since the definitions of ves, jes, etc. do not depend on the value of the coefficients, the relationship ves = jes + les + tes can be referred to as a "static identity". This is in contrast to so-called dynamic identities, an example of which is the following:

Define

```
VESPOS(X) =
    {'ie': '0.5(|A|+A)', 'je': '0.5(|B|+B)', 'ke': '0.5(|C|+C)',
     'ei': '0.5(|D|+D)', 'ej': '0.5(|E|+E)', 'ek': '0.5(|F|+F)',
     'ii': '0.5(|G|+G)', 'jj': '0.5(|H|+H)', 'kk': '0.5(|I|+I)',
     'ij': '0.5(|J|+J)', 'ik': '0.5(|K|+K)', 'ji': '0.5(|L|+L)',
     'jk': '0.5(|M|+M)', 'ki': '0.5(|N|+N)', 'kj': '0.5(|O|+O)',
     'ee': '0.5(|P|+P)'

     }
```

and

```
VESNEG(X) = X - VESPOS(X)
```

Keeping with our notation from above, we have

```
vespos(X) = ves(VESPOS(X))
```

This leads to the "dynamic" identity:

```
ves(X) = vespos(X) + vesneg(X)
```

Example: $Y = 1.5\overleftarrow{i} - 0.75\overleftarrow{j} + 0.75\overrightarrow{i} - 0.5\overrightarrow{j}$

```
ves(Y)= 1.5 - 0.75 + 0.75 - 0.5 = 1
vespos(Y) = 1.5 + 0.75 = 2.25
vesneg(Y) = -0.75 - 0.5 = -1.25
```

For any X, either

```
tespos(X) = 0
```

or

```
tesneg(X) = 0
```

(Why?) Of course, it does not follow from this property that tesposseq(X) or tesnegseq(X) is the zero sequence.

Often, what will happen is that `vesposseq(X)` *and* `vesnegseq(X)` *will be linear recurrence relations of the same order or higher . Although these sequences always seem to satisfy a recurrence relation, this is currently only a conjecture. The order of the recurrence relations can be quite high- usually 6th order or less but in rare cases up to 16.*

Lemma 2.3 *If* $X \in \mathcal{Z}^\infty$ *then* $\forall n : 4vespos(X^n)$ *is an integer (same for jespos, lespos, ...).*

Proof. $4vespos(X^n) = 4ibasepos(X^n) + 4jbasepos(X^n) + \cdots 4tespos(X^n)$ *and all the terms on the right side are integers by the definition of* $\mathcal{Z}^\infty$
Conjecture*: If* $X \in \mathcal{Z}^\infty$*, then* $vesposseq(X)$ *is a linear recurrence relation of order ??. (it follows immediately from this this that* $vesneg(X)$ *has similar properties).*

2.2 Using the Symbolic Multiplier

Let's look at how some of the above simple identities relate individual sequences. For this example, we define the floretions ELucI, and fib with the help of the Floretion Symbolic Multiplier:

```
ELucI =
{'ei': '.25', 'ke': '', 'ek': '', 'ej': '', 'kk': '.25', 'kj': '.25',
 'ki': '', 'ii': '.25', 'ee': '.25', 'ik': '', 'ij': '', 'ji': '', 'jj': '.25',
  'jk': '.25', 'je': '', 'ie': '.25'}

fib = {'ei': '', 'ke': '1', 'ek': '', 'ej': '', 'kk': '', 'kj': '', 'ki': '',
 'ii': '', 'ee': '', 'ik': '', 'ij': '', 'ji': '', 'jj': '', 'jk': '', 'je': '-1',
 'ie': '-1'}
```

The "seqFinder" command then gives

```
>>> seqFinder(T)
The first 15 terms of each sequence are as follows:
2jesseq: [-1, -1, -2, -3, -5, -8, -13, -21, -34, -55, -89, -144, -233,
-377, -610]
4jesposseq: [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
4jesnegseq: [-3, -2, -4, -6, -10, -16, -26, -42, -68, -110, -178, -288, -466, -754]
Note the dynamic identity jespos + jesneg = jes

4lesseq: [1, -1, 0, -1, -1, -2, -3, -5, -8, -13, -21, -34, -55, -89,
-144]
4lesposseq: [6, 5, 10, 15, 25, 40, 65, 105, 170, 275, 445, 720, 1165, 1885, 3050]
4lesnegseq: [-5, -6, -10, -16, -26, -42, -68, -110, -178, -288, -466, -754, -1220]
Note the dynamic identity lespos + lesneg = les

4tesseq: [1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843,
1364]

1vesseq: [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
2vesposseq: [4, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207]
2vesnegseq: [-4, -4, -7, -11, -18, -29, -47, -76, -123, -199, -322, -521, -843]
Note the dynamic identity vespos + vesneg = ves

4jesrightseq: [1, -2, -1, -3, -4, -7, -11, -18, -29, -47, -76, -123,
-199, -322, -521]
4jesrightposseq: [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
4jesrightnegseq: [0, -2, -1, -3, -4, -7, -11, -18, -29, -47, -76, -123, -199, -322]
Note the dynamic identity jesrightpos + jesrightneg = jesright

4jesleftseq: [-3, 0, -3, -3, -6, -9, -15, -24, -39, -63, -102, -165,
-267, -432, -699]

[snip]
```

The identity jes + les + tes = ves along with the above results now lead us to "conjecture":

```
2*Fibonacci(1,1) + Fibonacci(1,-1) = Lucas(1,3)
```

In this case it is very easy to get more sets of sequences which all generate their own set of identities, for example by slighty changing the floretions:

```
>>> T2 = Mult(ELucI,fib2)
Fibonacci(1,1) + Lucas(1,3) = 2*Fibonacci(1,2)

>>> T3 = Mult(EFibI,fib3)
Lucas(1,3) = 2*Fibonacci(0,1) + Fibonacci(1,1)

(vespos = jespos + lespos + tespos)
>>> T = Mult(ELucI,fib)
2*Lucas(4,7) = Lucas(3,4) + 5*Fibonacci(1,2)

>>> T2 = Mult(ELucI,fib2)
2*Fibonacci(3,5) = Lucas(1,3) + O2(5,7)

>>> T3 = Mult(EFibI,fib3)
2*Fibonacci(2,3) + O2(5,7) + Lucas(1,3) = 2*Fibonacci(5,8)
```

Of course, a computer does not spit out an entire sequence of numbers, but only finitely many terms. For example, it may return the numbers $(1, 2, 3, 4, 5, 6, \cdots)$ (choose $X = .5(\overleftarrow{i} + \overleftrightarrow{kk}) + \overleftrightarrow{ee})$, but at some point we have to terminate the program and "guess" that the sequence we see is in fact the natural numbers. Taking one of the above cases as an example, we wish to use the *static* identity $\mathrm{jes} + \mathrm{les} + \mathrm{tes} = \mathrm{ves}$ to automatically conclude that

```
2*Fibonacci(1,1) + Fibonacci(1,-1) = Lucas(1,3)
```

and not to spend time wondering whether "Lucas" really are the Lucas numbers and "Fibonacci" really are the Fibonacci numbers based on the n terms calculated. Now, it would always be possible to interpret the above as conjectures and prove the relations with other methods (ie using generating functions or simply comparing the first several terms, etc., where the latter method actually counts as a proof if sufficient initial terms are compared depending on the order of the recurrence). For static identities, Floret's Equation (see Basic Properties) proves that the sequences involved satisfy 4th order linear recurrences.

From a technical viewpoint, however, this does not carry over to dynamic identities– here, we only have the above conjecture. The reason is not that a given identity itself may be invalid, but that there is (currently) no direct way to prove that the sequences being generated really are (4th order or higher) linear recurrence relations. For example, the identity $\mathrm{vespos} = \mathrm{jespos} + \mathrm{lespos} + \mathrm{tespos}$ yields the relationship

```
2*Luc(4,7) = Luc(3,4) + 5*Fib(1,2)
```

from above, but by taken itself, this is still a conjecture.

2.3 Basic Properties

The following can be proved by multiplying out both sides of the equation. As there are many terms to calculate, this was done with the Symbolic Multiplier.

Proposition 2.4 ***Floret's Equation***

Any

$$
\begin{aligned}
X = {} & A\overleftarrow{i} + B\overleftarrow{j} + C\overleftarrow{k} + D\overrightarrow{i} + E\overrightarrow{j} + F\overrightarrow{k} + G\overleftrightarrow{ii} + H\overleftrightarrow{jj} + I\overleftrightarrow{kk} \\
& + J\overleftrightarrow{ij} + K\overleftrightarrow{ik} + L\overleftrightarrow{ji} + M\overleftrightarrow{jk} + N\overleftrightarrow{ki} + O\overleftrightarrow{kj} + P\overset{\leftrightarrows}{ee} \\
& = ibase(X)\overleftarrow{i} + jbase(X)\overleftarrow{j} + kbase(X)\overleftarrow{k} + basei(X)\overrightarrow{i} + basej(X)\overrightarrow{j} \\
& + basek(X)\overrightarrow{k} + ibasei(X)\overleftrightarrow{ii} + jbasej(X)\overleftrightarrow{jj} + kbasek(X)\overleftrightarrow{kk} \\
& + ibasej(X)\overleftrightarrow{ij} + ibasek(X)\overleftrightarrow{ik} + jbasei(X)\overleftrightarrow{ji} + jbasek(X)\overleftrightarrow{jk} \\
& + kbasei(X)\overleftrightarrow{ki} + kbasej(X)\overleftrightarrow{kj} + tes(X)\overset{\leftrightarrows}{ee}
\end{aligned}
$$

satisfies the following equation:

$$
\begin{aligned}
3 \cdot X^4 = {} & 12 \cdot P \cdot X^3 + \\
& (6 \cdot tes(X^2) - 24 \cdot P^2) \cdot X^2 + \\
& (32 \cdot P^3 - 24 \cdot P \cdot tes(X^2) + 4 \cdot tes(X^3)) \cdot X + \\
& (3 \cdot tes(X^4) - 6 \cdot (tes(X^2))^2 - 16 \cdot P \cdot tes(X^3) + 48 \cdot P^2 \cdot tes(X^2) - 32 \cdot P^4) \cdot ee
\end{aligned}
$$

Corollary 2.5 *For any X, $tesseq(X)$ satisfies a 4th order linear recurrence sequence.*

Proof Use Floret's Equation along with the property that tes() is a linear functional.

Corollary 2.6 *For any X*

- *If $tes(X) = tes(X^2) = tes(X^3) = tes(X^4) = 0$, then $X^m = 0\ \forall m \geq 4$.*

- *If $tes(X) = tes(X^2) = tes(X^3) = 0$, then $X^{4m} = tes(X^{4m})\overset{\leftrightarrows}{ee}\ \forall m > 1$.*

Example 2.7 *For an example of $tes(X) = tes(X^2) = tes(X^3) = 0$, choose $X = \overleftarrow{i} + \overleftrightarrow{ii}$*

Proposition 2.8 *If tesseq(X) is a sequence of integers, then*

$$p \mid tes(X^p) - tes(X)^p$$

for all odd primes.

The following lemma is almost trivial to mention, but plays a crucial role in the proof.

Lemma 2.9 *Let G be any group and choose $g_1, g_2, ...g_m \in G$.*

Then from $(g_1) \cdot (g_2) \cdot \cdots \cdot (g_m) = u$ where u is the identity element, it follows that $(g_2) \cdot \cdots \cdot (g_m) \cdot (g_1) = u$

Proof $(g_2) \cdot \cdots \cdot (g_m)$ is the unique inverse of g_1 and therefore commutes with g_1 by the definition of a group. q.e.d.

Proof of proposition:

Let p be an odd prime and

$$X = A\overleftarrow{i} + B\overleftarrow{j} + C\overleftarrow{k} + D\overrightarrow{i} + E\overrightarrow{j} + F\overrightarrow{k} + G\overleftrightarrow{ii} + H\overleftrightarrow{jj} + I\overleftrightarrow{kk}$$
$$+ J\overleftrightarrow{ij} + K\overleftrightarrow{ik} + L\overleftrightarrow{ji} + M\overleftrightarrow{jk} + N\overleftrightarrow{ki} + O\overleftrightarrow{kj} + P\overleftrightarrow{ee}$$

How many terms belong to tes(X^p), i.e. the coefficient of the identity $\overleftrightarrow{ee}$ from the expanded product X^p.

Apparently, the answer is 16^{p-1} since we have 16 choices from each set of parenthesis... except the last set (why?). Now, many of those terms will cancel out and determining which terms have which signs could get extremely complicated (the LEMMA will be used to get around the problem).

Disregarding signs, any such term may be seen as a string with p letters.

Assume, for example, that -ADDB...ACMP is one of the actual terms from tes(X^p). It follows from the distributive law that rotating this string as a bracelet to form PADDB...ACM must also be a term in the product. The sign of this term is now critical to the proof since it could either double or cancel out the term before rotation. However, since the base vectors 'i, 'j, ... 'ee' can be seen as elements of a group (see Structure of the Floretion Group (R. Mathar)) the above lemma applies, and we conclude the sign of this term must likewise be negative.

So in our running example, we know there must be terms:

```
ADDB...ACMP
PADDB...ACM
MPADDB...AC,
etc.
```

in the expansion of tes(X^p) which all have the same (negative) sign. Since p is prime > 2, the number of such terms must be p (see Wikipedia, *Necklace Proof of Fermat's Little Theorem* and, if you wish, consider why it doesn't work for $p = 2$). Thus, the terms in the expansion of tes(X^p) are partioned into groups and the number of terms in each group is p - with the exception of the single term tes$(X)^p$, which is subtracted off. q.e.d.

Note: For p = 3 we have tes(X^3):

```
+3.0PHH +6.0LBD -3.0PFF +3.0PMM +6.0DCN -3.0CCP -6.0GOM +3.0NNP +6.0BMF
-3.0PAA -3.0DPD +6.0OEC +6.0KAF +6.0NMJ +6.0HEB -3.0BBP +6.0LKO -6.0NKH
+6.0JAE -6.0LJI +6.0IGH +3.0PJJ +3.0POO +1.0PPP +3.0PII +6.0AGD +6.0FCI
+3.0PGG +3.0PLL -3.0EEP +3.0KKP
```

Proposition 2.10 *Let* X *and* $Y = y_0 \overleftarrow{i} + y_1 \overleftarrow{j} + y_2 \overleftarrow{k} + y_3 \overrightarrow{i} + y_4 \overrightarrow{j} + y_5 \overrightarrow{k} + y_6 \overleftrightarrow{ii} + y_7 \overleftrightarrow{jj} + y_8 \overleftrightarrow{kk} + y_9 \overleftrightarrow{ij} + y_{10} \overleftrightarrow{ik} + y_{11} \overleftrightarrow{ji} + y_{12} \overleftrightarrow{jk} + y_{13} \overleftrightarrow{ki} + y_{14} \overleftrightarrow{kj} + y_{15} \overleftrightarrow{ee}$ *be any floretions. Then tesseq*$[X \cdot Y] =$ *tesseq*$[Y \cdot X]$

Proof

We have
$\text{tes}(X \cdot Y) = -\sum_{i=0}^{5} x_i y_i + \sum_{i=6}^{15} x_i y_i = -\sum_{i=0}^{5} y_i x_i + \sum_{i=6}^{15} y_i x_i = \text{tes}(Y \cdot X)$

Define $Z = X \cdot (Y \cdot X)^{(n-1)}$, then
$\text{tesseq}[X \cdot Y](n) = \text{tes}((X \cdot Y)^n) = \text{tes}((X \cdot (Y \cdot X)^{(n-1)}) \cdot Y) = \text{tes}(Z \cdot Y)$
$= \text{tes}(Y \cdot Z) = \text{tes}(Y \cdot (X \cdot (Y \cdot X)^{(n-1)})) = \text{tes}((Y \cdot X) \cdot (Y \cdot X)^{(n-1)}) = \text{tesseq}[Y \cdot X](n)$

q.e.d.

Surprisingly perhaps, despite the above lemma, there is still no way in general to deduce from the above that $\text{tes}(XYXY) = \text{tes}(X^2Y^2)$. Thus if $X = Y + Z$, then the binomial theorem $\text{tes}(X^n) = \sum_{k=0}^{n} \binom{n}{k} \text{tes}(Y^n Z^{n-l})$ does **not** hold in general. It may seem strange that this apparently comes in many cases, along with the fact that there exist many zero divisors in the space, as a clear advantage! For example, we can view the strings $XYXY$ and $XXYY$ as necklaces which are "not equivalent under tes" (see Chapter: Necklaces). The property that there exist zero divisors will result in some necklaces being thrown out. As a result, rules such as those given for this sequence: A113166: Total number of white pearls remaining in chest (which equals the Fibonacci numbers for prime n- thanks to M. Alekseyev and A.Karttunen) will not appear ad hoc, but as a direct result of the above considerations.

Note that if $(a_n)_{n \in \mathbb{N}}$ is a sequence of integers, then the *binomial transform* of $(a_n)_{n \in \mathbb{N}}$ is defined as $\sum_{k=0}^{n} \binom{n}{k} a_n$

In the following theorem, it makes no difference if "tes" is replaced by "ibase", etc..

Theorem 2.11 *If* $a' = (1, a_0, a_1, a_2, ...) = (1, tes(X), tes(X^2), tes(X^3), ...)$, *then* $b' = (1, b_0, b_1, b_2, ...) = (1, tes(X + \overleftrightarrow{ee}), tes((X + \overleftrightarrow{ee})^2), tes((X + \overleftrightarrow{ee})^3), ...)$ *is the binomial transform of* a'.

Proof $\text{tes} : X \mapsto \text{tes}(X)$ is a linear mapping. Moreover, the unit ee commutes with any X. Therefore,

$\sum_{k=0}^{n} \binom{n}{k} a'_n = \sum_{k=0}^{n} \binom{n}{k} \text{tes}(X^n) = \text{tes}(\sum_{k=0}^{n} \binom{n}{k} X^n) = \text{tes}((X + \overleftrightarrow{ee})^n) = b'_n$

q.e.d.

Just for fun: We know that the binomial transform of the Fibonacci numbers is equal to its bisection. Assume $(1, \text{tes}(X), \text{tes}(X^2), \text{tes}(X^3), ...)$ is the Fibonacci numbers starting at 1. Then by definition, the bisection of this sequence is $(1, \text{tes}(X^2), \text{tes}(X^4), ...)$. Is there a conflict here between this bit of information and the above theorem?

Corollary 2.12 *For any* $m \in \mathbb{Z}$, *tesseq*$[X]$ *is an integer sequence if and only if tesseq*$[X + m \cdot \overleftrightarrow{ee}]$ *is as well.*

Remark 2.13 *Another way to write what we have shown above is*

$$X \in \mathcal{Z}^\infty \Leftrightarrow X \pm \overleftrightarrow{ee} \in \mathcal{Z}^\infty$$

and several somewhat thorny conjectures, etc. to come could be simplified if we had a generalized version of the above:

$$X \in \mathcal{Z}^\infty \Leftrightarrow X \pm \overleftarrow{i} \in \mathcal{Z}^\infty \Leftrightarrow X \pm \overleftarrow{j} \in \mathcal{Z}^\infty \Leftrightarrow X \pm \overleftarrow{k} \in \mathcal{Z}^\infty (\dots)$$

However, this requires a bit more machinery due to the fact that $\overleftarrow{i}$ *does not, in general, commute with* X. *The generalized version is stated in Corollary .*

Remark 2.14 *Here in and in the sequel, we shall misuse notation by identifying* $(a, b, c) \in \mathbb{R}^3$ *with a pure quaternion* $a\overleftarrow{i} + b\overleftarrow{j} + c\overleftarrow{k}$ *or, as in 2.16, with a pure floretion* $aX_1 + bX_2 + cX_3$. *By "pure floretion", we mean* $TES(X) = 0$ *or, equivalently,* $X = JES(X) + LES(X)$.

Lemma 2.15 ***Vector Product, Dot Product and Determinant, Pure Quaternion Version***
Let (a, b, c), (d, e, f), $(g, h, i) \in \mathbb{R}^3$ *be 3 vectors and* $a\overleftarrow{i} + b\overleftarrow{j} + c\overleftarrow{k}$, $d\overleftarrow{i} + e\overleftarrow{j} + f\overleftarrow{k}$, $g\overleftarrow{i} + h\overleftarrow{j} + i\overleftarrow{k}$. *their "pure quaternion" counterparts. Define.*

$$W = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We have the following properties:

$$(a,b,c)\textit{scalar-product }(d,e,f) = -\textit{tes}((a\overleftarrow{i} + b\overleftarrow{j} + c\overleftarrow{k}) \cdot (d\overleftarrow{i} + e\overleftarrow{j} + f\overleftarrow{k})) \tag{1}$$

$$(a,b,c)\textit{vector-product }(d,e,f) = \textit{JES}((a\overleftarrow{i} + b\overleftarrow{j} + c\overleftarrow{k}) \cdot (d\overleftarrow{i} + e\overleftarrow{j} + f\overleftarrow{k})) \tag{2}$$

$$\textit{det}(W) = -\textit{tes}((a\overleftarrow{i} + b\overleftarrow{j} + c\overleftarrow{k}) \cdot (d\overleftarrow{i} + e\overleftarrow{j} + f\overleftarrow{k}) \cdot (g\overleftarrow{i} + h\overleftarrow{j} + i\overleftarrow{k})) \tag{3}$$

Proposition 2.16 ***Floretion Vector Product***

For any 2 floretions $aX + bY + cZ$ and $dX + eY + fZ$ where $X, Y, Z \in \{\overleftarrow{i}, \overleftarrow{j}, \overleftarrow{k}, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}, \overleftrightarrow{ii}, \overleftrightarrow{jj}, \overleftrightarrow{kk}, \overleftrightarrow{ij}, \overleftrightarrow{ik}, \overleftrightarrow{ji}, \overleftrightarrow{jk}, \overleftrightarrow{ki}, \overleftrightarrow{kj}\}$ are pairwise different base floretions:

If $(X + Y + Z)^2 = TES((X + Y + Z)^2)$, then

$$(a,b,c)\textit{vector-product }(d,e,f) = (\textit{JES} + \textit{LES})((aX + bY + cZ) \cdot (dX + eY + fZ)) \tag{4}$$

Example: take $X_1 = \overleftrightarrow{ii}, X_2 = \overleftrightarrow{ij}, X_3 = \overleftrightarrow{ik}$.

Proof.
Assume $(X+Y+Z)^2 = \text{TES}((X+Y+Z)^2)$. Then (JES + LES)$((aX + bY + cZ) \cdot (dX + eY + fZ))$ = (JES + LES)$((aeXY + bdYX + afXZ + cdZX + bfYZ + ceZY)$. Since X, Y and Z are base floretions, we know that either $XY = YX$ or $XY = -YX$. Assume $XY = YX$, then $(X + Y + Z)^2 = 2XY + \cdots$. Since XY can not be the unit vector and also can not be equal to $\pm YZ$ or $\pm XZ$, we have $(X + Y + Z)^2 \neq \text{TES}((X + Y + Z)^2)$, a contradiction. Since JES and LES are (linear) projection operators, it follows that (JES + LES)$((aX+bY+cZ)\cdot(dX+eY+fZ))$ = (JES + LES)$(((ae-bd)XY)+$ (JES + LES)$(((af - cd)XZ)$ + (JES + LES)$(((bf - ce)YZ)$, q.e.d.

Determinants have the important relationship $\det(AB) = \det(A) \cdot \det(B)$. For this reason, the following result may be particularly interesting.

Proposition 2.17 *Let X_1, X_2, X_3 be 3 floretions with the property $X_i = JESLEFT(X_i)$ and X_4, X_5, X_6 have the property $X_i = JESRIGHT(X_i)$.*

Then

$$\textit{tes}(X_1 \cdot X_2 \cdot X_3) \cdot \textit{tes}(X_4 \cdot X_5 \cdot X_6) = \textit{tes}(X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5 \cdot X_6)$$

Proof. Use the Floretion Symbolic Multiplier.

Corollary 2.18 *Let X_1, X_2, X_3 be 3 floretions with the property $X_i = JESLEFT(X_i)$ and X_4, X_5, X_6 have the property $X_i = JESRIGHT(X_i)$.*

Then

$$tes((X_1 \cdot X_2 \cdot X_3 + X_4 \cdot X_5 \cdot X_6)^2) = tes((X_1 \cdot X_2 \cdot X_3)^2) + tes((X_4 \cdot X_5 \cdot X_6)^2) + 2tes(X_1 \cdot X_2 \cdot X_3) \cdot tes(X_4 \cdot X_5 \cdot X_6)$$

Determinants have the important relationship $\det(AB) = \det(A) \cdot \det(B)$. For this reason, the following result may be particularly interesting.

Proposition 2.19 *Let X_1, X_2, X_3 be 3 floretions with the property $X_i = JESLEFT(X_i)$ and X_4, X_5, X_6 have the property $X_i = JESRIGHT(X_i)$.*

Then

$$tes(X_1 \cdot X_2 \cdot X_3) \cdot tes(X_4 \cdot X_5 \cdot X_6) = tes(X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5 \cdot X_6)$$

Proof. Use the Floretion Symbolic Multiplier.

For several of the following proposition(s), it makes no difference whether X is a floretion which generates a 2nd, 3rd, or 4th order recurrence relation. In fact, it is totally independent of the form which X might take. As with static identities, this is perhaps interesting on a deeper level, for it means that 1st, 2nd, 3rd, or 4th order sequences all have the proposition in common.

Proposition 2.20 ***"Wave Equation"****. In the following, let X and Y be any floretions.*

$$X = JES(X) \Rightarrow X^{2n} = LES(X^{2n}) + TES(X^{2n}) \quad (5)$$

$$X = JES(X) \Rightarrow X^{2n+1} = JES(X^{2n+1}) \quad (6)$$

$$X = LES(X) + TES(X) \Rightarrow \forall m \in \mathbb{N} : X^m = LES(X^m) + TES(X^m) \quad (7)$$

Also:

$$JES(X^2) = 2JES(X \cdot JES(X)) \quad (8)$$

Proof. The above statements are all easily shown with the symbolic multiplier. For example,

```
>>>Sub(JES(Mult(X,X)),  Add(JES(Mult(X, JES(X))),JES(Mult(X, JES(X)))))
```

which returns an empty dictionary, proves the last statement.

As there are an infinite number of different floretions which generate Fibonacci numbers or Pell numbers, one might wonder whether it is possible to generate a 2nd order linear recurrence using "pure" floretions $X = A\overleftarrow{i} + B\overleftarrow{j} + C\overleftarrow{k} + D\overrightarrow{i} + E\overrightarrow{j} + F\overrightarrow{k}$. A few simple corollaries to 2.20 demonstrate why this is not possible.

Corollary 2.21 $X = JES(X) \Rightarrow$

```
tesseq(X)(2n+1) = tes(X^{2n+1})= 0
ibaseseq(X)(2n+2) = ibaseseq(X^{2n+2})= 0
```

Corollary 2.22 *If $X = JES(X)$ and $aX + bX^2 = X^3$ with $aX \neq 0$, then $b = 0$.*

Proof. From $X = \text{JES}(X)$ it follows that $X^2 = \text{LES}(X^2) + \text{TES}(X^2)$ and $X^3 = \text{JES}(X^3)$. By definition of JES, LES and TES, it follows that $bX^2 = 0$. Assume $X^2 = 0$, then $X^3 = X^2 \cdot X = 0$ and thus $aX = 0$, a contradition. It follows that $b = 0$, q.e.d

Corollary 2.23 $JES(X^2) = (JES(X))^2 \Leftrightarrow X = 0$

Lemma 2.24

$$X = JESLEFT(X) + TES(X) =: \mathcal{I}$$
$$Y = JESRIGHT(Y) + LES(Y) =: \mathcal{II}$$

$$\text{If } \mathcal{I} \text{ and } \mathcal{II} \Rightarrow XY = JESRIGHT(X \cdot Y) + LES(X \cdot Y) \tag{9}$$

$$\mathcal{I} \text{ and } \mathcal{II} \Rightarrow YX = JESRIGHT(Y \cdot X) + LES(Y \cdot X) \tag{10}$$

$$\mathcal{I} \text{ and } \mathcal{II} \Rightarrow JESRIGHT(X \cdot Y) = JESRIGHT(Y \cdot X) \tag{11}$$

Lemma 2.25 *If $X = A\overleftarrow{i} + B\overleftarrow{j} + C\overleftarrow{k} + D\overrightarrow{i} + E\overrightarrow{j} + F\overrightarrow{k} + G\overleftrightarrow{ii} + H\overleftrightarrow{jj} + I\overleftrightarrow{kk} + J\overleftrightarrow{ij} + K\overleftrightarrow{ik} + L\overleftrightarrow{ji} + M\overleftrightarrow{jk} + N\overleftrightarrow{ki} + O\overleftrightarrow{kj} + P\overleftrightarrow{ee}$, then $ibase(X^2)$, $jbase(X^2)$, etc. with the exception of $tes(X^2)$ can always be written in the form of 2 times the sum of 4 terms (where each of the 4 terms is in the form of a product of two different base coefficients of X).*

Proof. Using the Floretion Symbolic Multiplier:

```
>>> Mult(X,X)
ee   +1.0JJ +1.0PP -1.0AA +1.0LL -1.0FF +1.0II +1.0NN
      +1.0KK -1.0EE +1.0MM +1.0GG +1.0OO -1.0BB +1.0HH -1.0CC -1.0DD
ie   -2.0FK -2.0GD -2.0EJ +2.0PA
je   -2.0MF +2.0BP -2.0DL -2.0HE
ke   +2.0CP -2.0FI -2.0EO -2.0DN
ei   -2.0LB +2.0PD -2.0CN -2.0GA
ej   -2.0AJ -2.0OC -2.0HB +2.0EP
ek   +2.0PF -2.0KA -2.0MB -2.0CI
```

```
ii   +2.0HI -2.0OM +2.0DA +2.0PG
jj   -2.0NK +2.0PH +2.0EB +2.0IG
kk   +2.0FC -2.0LJ +2.0IP +2.0GH
ij   +2.0MN +2.0AE -2.0LI +2.0JP
ik   +2.0OL -2.0HN +2.0KP +2.0AF
ji   +2.0KO +2.0BD -2.0JI +2.0PL
jk   +2.0JN +2.0BF -2.0GO +2.0MP
ki   -2.0KH +2.0MJ +2.0NP +2.0DC
kj   +2.0KL -2.0MG +2.0OP +2.0EC
```

Proposition 2.26 *(proof incomplete- so still only a conjecture!) If the fractional parts of the base coefficients of X are all in the set* $\{0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}\}$ *then* $X \in \mathcal{Z}^\infty \Leftrightarrow 4tesseq(X)$ *is a sequence of integers*

Note: *base coefficient of X* is abbreviated B.C. of X in the proof below

Proof

"$\Rightarrow$" follows immediately from the definition of the set $\mathcal{Z}^\infty$.

"$\Leftarrow$"
Assume $X \notin \mathcal{Z}^\infty$. We show 4tesseq($X$) is not a sequence of integers

Claim I: Let x_i be a B.C.of X (thus, $x_i = \text{ibase}(X)$ or $x_i = \text{jbase}(X)$, etc). If there is an odd number of B.C.'s of X with the property that $0 < [|x_i|] < \frac{1}{4}$ (where $[|x_i|]$ denotes the fractional part of the absolute value of x_i) then 4tesseq(X) is not an integer sequence.

Proof of Claim I: Since the denominators must be powers of two, a B.C. of X whose fractional part is "greater than zero but less than one-fourth" must have a denominator which is at least 2^3 (and an odd numerator). Since there are an odd number of such basis vectors, the number

$$\text{tes}(X^2) = -\sum_{i=0}^{5} x_i^2 + \sum_{i=6}^{15} x_i^2$$

cannot equal zero and its fractional part must be less than or equal to $\frac{\alpha}{64}$ where α is an odd integer. It follows that 4tes(X^2) is not an integer. Hence, 4tesseq(X) cannot be a sequence of integers. q.e.d. (Claim I)

Claim II: If there are exactly two B.C.'s x_1, x_2 of X with the property that $0 < [|x_i|] < \frac{1}{4}$ then 4tesseq(X) is not an integer sequence.

Proof of Claim II: We again examine all the B.C.'s of X^2:

```
>>> Mult(X,X)
ee   +1.0JJ +1.0PP -1.0AA +1.0LL -1.0FF +1.0II +1.0NN
      +1.0KK -1.0EE +1.0MM +1.0GG +1.0OO -1.0BB +1.0HH -1.0CC -1.0DD
ie   -2.0FK -2.0GD -2.0EJ +2.0PA
je   -2.0MF +2.0BP -2.0DL -2.0HE
ke   +2.0CP -2.0FI -2.0EO -2.0DN
ei   -2.0LB +2.0PD -2.0CN -2.0GA
ej   -2.0AJ -2.0OC -2.0HB +2.0EP
ek   +2.0PF -2.0KA -2.0MB -2.0CI
ii   +2.0HI -2.0OM +2.0DA +2.0PG
jj   -2.0NK +2.0PH +2.0EB +2.0IG
kk   +2.0FC -2.0LJ +2.0IP +2.0GH
ij   +2.0MN +2.0AE -2.0LI +2.0JP
ik   +2.0OL -2.0HN +2.0KP +2.0AF
ji   +2.0KO +2.0BD -2.0JI +2.0PL
jk   +2.0JN +2.0BF -2.0GO +2.0MP
ki   -2.0KH +2.0MJ +2.0NP +2.0DC
kj   +2.0KL -2.0MG +2.0OP +2.0EC
```

The product $\pm 2x_1 \cdot x_2$ appears in at most one of the above terms. Just for arguments sake, we may assume $x_1 =$ F and $x_2 =$ B. Then the product $x_1 \cdot x_2 =$ F $\cdot$ B appears (twice) in jbasek$(X^2) = +2.0\text{JN} + 2.0\text{BF} - 2.0\text{GO} + 2.0\text{MP}$. It follows that there is exactly one basis vector coefficient of X^2 (namely jbasek(X^2)) with the property $0 < [|\text{jbasek}(X^2)|] < \frac{1}{4}$. Now Claim I applies.

If $x_1 =$ A and $x_2 =$ B then $4tes(X^2)$ cannot be an integer since in this case both x_1 and x_2 belong to JES(X) and the terms cannot cancel each other's fractional parts out when they appear as a sum of squares in tes(X^2) .

Seqfan Question: An "anomoly" occurs here when $x_1 =$ A and $x_2 =$ H or $x_2 =$ I . Note that $Y^2 = 0$ with $Y = \overleftarrow{i} + \overleftrightarrow{jj}$. If $[A] = [H]$, these two terms do cancel each other's fractional parts out in tes(X^2). At this point we can construct a floretion which *appears* to contradict the entire proposition itself. For example $Y = \frac{1}{8}\overleftarrow{i} + 4\overleftarrow{j} + 4\overleftrightarrow{ii} + \frac{1}{8}\overleftrightarrow{jj}$. Then $Y^2 = -\overrightarrow{i} - \overrightarrow{j} + \overleftrightarrow{kk}$, $Y^3 = 4\overleftarrow{i} + \frac{1}{8}\overleftarrow{j} - \frac{1}{8}\overleftrightarrow{ij} - 4\overleftarrow{ji}$ and $Y^4 = -\overleftrightarrow{ee}$. The question in this case is "how did we get here in the first place?". Indeed, since the B.C.'s of $4X$ are all integers by definition (see introduction), we must have $X^m = Y$ for some $m > 1$ and X where the B.C.'s of $4X$ are all integers to produce a genuine counterexample in this case. Obviously, setting $X = Y$ (since $Y^9 = Y$) does not pass the condition that the B.C.'s of $4X$ must be integers.

I see two ways out of the situation: the first, of course, would be to show that there are no such counterexamples. The second -gulp- would be to change the proposition itself. Since there are several corollaries which depend on this proposition, it had better be something minor. Perhaps something like (whether it is a major or minor change is currently unclear!):

If the fractional parts of the base coefficients of X are all in the set $\{0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}\}$ then $X^2 \in \mathcal{Z}^\infty \Leftrightarrow$ 4tesseq(X) is a sequence of integers

q.e.d. (Claim II)

Claim III: If there are exactly four B.C.'s x_1, x_2, x_3, x_4 of X with the property that $0 < [|x_i|] < \frac{1}{4}$ then 4tesseq(X) is not an integer sequence.

Proof of Claim III: Exactly two of the basis vector coefficients x_1, x_2, x_3, x_4 must belong to JES(X) and two must belong to LES(X) (consider tes(X^2) to see why and note that if one of the basis vectors belongs to TES(X) there is nothing to prove).

Assume the first two belong to JES(X) and the second two to LES(X). If we look at the basis vectors of X^2, we conclude that the product $x_1 \cdot x_2$ is nowhere to be found! We are only left with the combinations

- $x_1 \cdot x_3$
- $x_1 \cdot x_4$
- $x_2 \cdot x_3$
- $x_2 \cdot x_4$
- $x_3 \cdot x_4$

If $x_3 = tes(X) = P$ or $x_4 = tes(X) = P$ there is nothing to prove. If the above 5 products belong to 5 different basis vectors, we can use Claim I. We are left to consider what happens when the above products belong to exactly 4 different basis vectors. Can $x_1 \cdot x_3$ and $x_1 \cdot x_4$ belong to the same basis vector? No (check). The only combinations are that $x_1 \cdot x_3$ and $x_2 \cdot x_4$ belong to one basis vector or that $x_1 \cdot x_4$ and $x_2 \cdot x_3$ belong to one basis vector. Assume $x_1 \cdot x_3$ and $x_2 \cdot x_4$ belong to one basis vector. Again, if we look at the basis vectors of X^2 and take into account that both x_1 and x_2 belong to JES(X), then we see that there is no such possibility unless either $x_3 = P$ or $x_4 = P$. This proves Claim III.

The next claim should be no surprise: **Claim** IV: If there are exactly six B.C.'s x_1, x_2, x_3, x_4 of X with the property that $0 \leq [|x_i|] \leq \frac{1}{4}$ then 4tesseq(X) is not an integer sequence.

Proof of Claim IV: Exactly three basis vectors x_1, x_2, x_3 must belong to JES(X) and three must belong to LES(X). In this case, the products $x_1 \cdot x_2$, $x_2 \cdot x_3$ and $x_3 \cdot x_1$ are nowhere to be found in the basis vectors of X^2. Thus, we are again left with an odd number of final choices...

q.e.d.

We can now prove the following two corollaries. They follow from the last proposition and Proposition 2.10:

Corollary 2.27 $X \cdot Y \in \mathcal{Z}^\infty$, *then* $Y \cdot X \in \mathcal{Z}^\infty$

Proof. This simple proof is left as an exercise.

This corollary has its own corollary:

Corollary 2.28

$$X \in \mathcal{Z}^\infty \Leftrightarrow X \pm \overleftarrow{i} \in \mathcal{Z}^\infty \Leftrightarrow X \pm \overleftarrow{j} \in \mathcal{Z}^\infty \Leftrightarrow X \pm \overleftarrow{k} \in \mathcal{Z}^\infty (\dots) \tag{12}$$

Proof. Let $X \in \mathcal{Z}^\infty$ and choose $m \in \mathbb{N}$. It follows by definition that 4 times each base coefficient of X^m is an integer and a quick inspection reveals that 4 times the base coefficients of $X^m \overleftarrow{i}$ must also be integers. Thus,

$$X \in \mathcal{Z}^\infty \Leftrightarrow X \cdot \overleftarrow{i} \in \mathcal{Z}^\infty$$

Multiplying out the expression $(X + \overleftarrow{i})^m$ gives 2^m terms of the form

$$X \cdot X \cdot X \overleftarrow{i} \cdot \overleftarrow{i} \cdot X \cdot \overleftarrow{i} \cdot X \cdots X \cdot \overleftarrow{i} \cdot X$$

If we can show that 4 times the base coefficients of any such term is an integer, we are finished. With no loss to generality, the same example is used. We show

$$X \cdot X \cdot X \overleftarrow{i} \cdot \overleftarrow{i} \cdot X \cdot \overleftarrow{i} \cdot X \cdots X \cdot \overleftarrow{i} \cdot X \in \mathcal{Z}^\infty$$

Since $\overleftarrow{i} \cdot \overleftarrow{i} = -\overleftrightarrow{ee}$, we only need show

$$X \cdot X \cdot X \cdot X \cdot \overleftarrow{i} \cdot X \cdots X \cdot \overleftarrow{i} \cdot X \in \mathcal{Z}^\infty$$

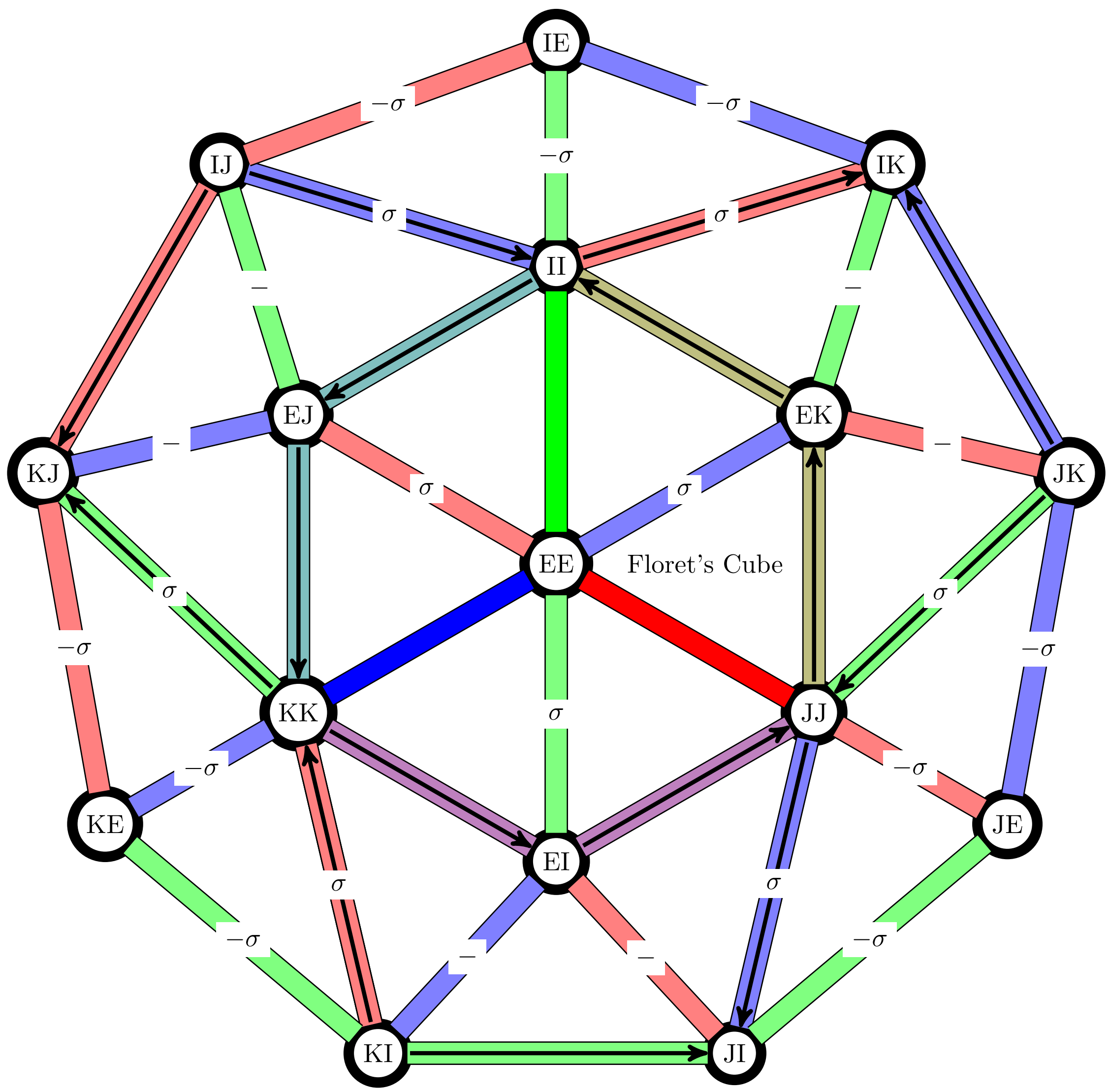
By Corollary 2.27,

$$\begin{aligned}
&(X \cdot X \cdot X \cdot X \cdot \overleftarrow{i} \cdot X \cdots X \cdot \overleftarrow{i}) \cdot X \in \mathcal{Z}^\infty \\
&\Leftrightarrow X \cdot (X \cdot X \cdot X \cdot X \cdot \overleftarrow{i} \cdot X \cdots X \cdot \overleftarrow{i}) \in \mathcal{Z}^\infty \\
&\Leftrightarrow (X \cdot X \cdot X \cdot X \cdot X \cdot \overleftarrow{i} \cdot X \cdots X) \cdot \overleftarrow{i} \in \mathcal{Z}^\infty \\
&\Leftrightarrow X \cdot X \cdot X \cdot X \cdot X \cdot \overleftarrow{i} \cdot X \cdots X \in \mathcal{Z}^\infty \\
&\Leftrightarrow X \cdot X \cdot X \cdot X \cdot X \cdot \overleftarrow{i} \cdot (X \cdots X) \in \mathcal{Z}^\infty \\
&\Leftrightarrow (X \cdots X) \cdot (X \cdot X \cdot X \cdot X \cdot X \cdot \overleftarrow{i}) \in \mathcal{Z}^\infty \\
&\Leftrightarrow (X \cdots X \cdot X \cdot X \cdot X \cdot X \cdot X) \cdot \overleftarrow{i} \in \mathcal{Z}^\infty \\
&\Leftrightarrow X \cdots X \cdot X \cdot X \cdot X \cdot X \cdot X \in \mathcal{Z}^\infty
\end{aligned}$$

The last statement immediately follows from $X \in \mathcal{Z}^\infty$ q.e.d.

3 Floret's Cube

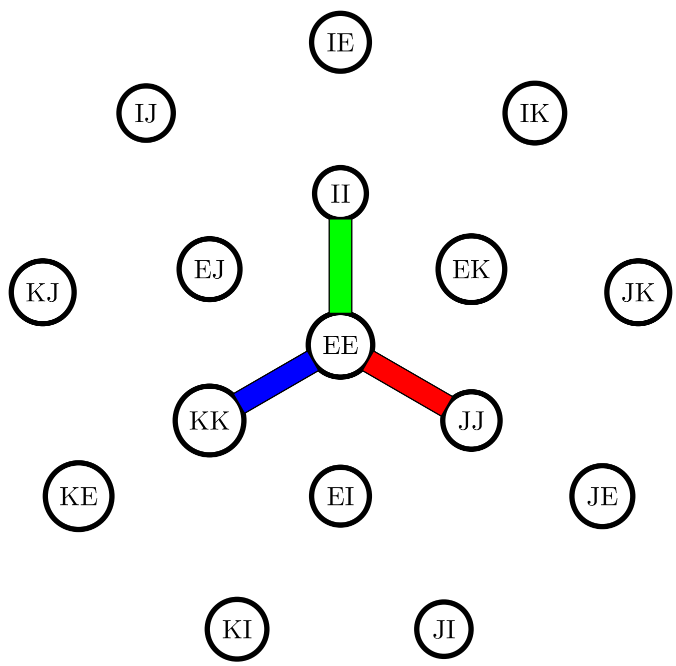
Laxly stated, with so many possibilities... how do we know which floretions are the interesting ones? ***Floret's Cube*** can be used to make many educated guesses in that regard:



The meaning of the symbols of Floret's Cube are explained at the end of this chapter.

3.1 Power Sequences

$$F = 0.25(II + JJ + KK + EE)$$



Theorem 3.1 *(Power Sequences)*

- *Let* $F = 0.25(ii + jj + kk + ee)$ *If* X *is any floretion, then* $tes((Fx)^n) = tes((xF)^n) = (ibasei(x) + jbasej(x) + kbasek(x) + tes(x))^n = (x_6 + x_7 + x_8 + x_{15})^n = (tes(Fx))^n$. *The same results hold for ibasei, jbasej and kbase k, i.e.* $ibasei((Fx)^n) = (x_6 + x_7 + x_8 + x_{15})^n$

- $F_{ii} = 0.25(ii - jj - kk + ee) \Rightarrow tes((F_{ii}x)^n) = (x_6 - x_7 - x_8 + x_{15})^n$

- $F_{jj} = 0.25(-ii + jj - kk + ee) \Rightarrow tes((F_{jj}x)^n) = (-x_6 + x_7 - x_8 + x_{15})^n$

- $F_{kk} = 0.25(-ii - jj + kk + ee) \Rightarrow tes((F_{kk}x)^n) = (-x_6 - x_7 + x_8 + x_{15})^n$

- *Let* $G = 0.25(-\overleftarrow{i} - \overrightarrow{i} + jk + kj)$*. If* X *is any floretion, then* $tes((Gx)^n) = (x_{12} + x_{14} + x_0 + x_3)^n$

- *Let* $G_{ii} = 0.25(\overleftarrow{i} + \overrightarrow{i} + jk + kj)$*. If* X *is any floretion, then* $tes((G_{ii}x)^n) = (x_{12} + x_{14} - x_0 - x_3)^n$

Outline of proof (one of several possible). Use induction and the properties $0 = (\overrightarrow{i} + jk)F = (\overrightarrow{j} + ki)F = (\overrightarrow{k} + ij)F = (\overleftarrow{i} + kj)F = (\overleftarrow{j} + ik)F = (\overleftarrow{k} + ji)F$ and $iiF = jjF = kkF = F$.

Definition 3.2 *(*σ *and* ς *operators)*
The σ *operator is defined to "reverse the arrows" of basis vectors:*

$$\begin{aligned}\sigma X &= basei(X)\overleftarrow{i} + basej(X)\overleftarrow{j} + basek(X)\overleftarrow{k} + ibase(X)\overrightarrow{i} + jbase(X)\overrightarrow{j} + kbase(X)\overrightarrow{k} \\ &+ ibasei(X)\overleftrightarrow{ii} + jbasej(X)\overleftrightarrow{jj} + kbasek(X)\overleftrightarrow{kk} + jbasei(X)\overleftrightarrow{ij} + kbasei(X)\overleftrightarrow{ik} + ibasej(X)\overleftrightarrow{ji} \\ &+ kbasej(X)\overleftrightarrow{jk} + ibasek(X)\overleftrightarrow{ki} + jbasek(X)\overleftrightarrow{kj} + tes(X)\overleftrightarrow{ee}\end{aligned}$$

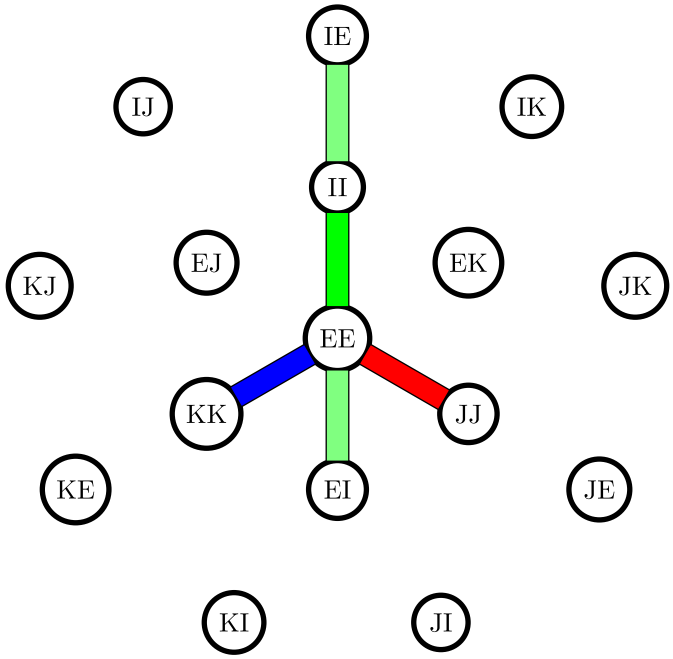
Define $F_\sigma = 0.5(ii + jj + kk - ee)$*. A check reveals the interesting property that* $\sigma X = F_\sigma X F_\sigma$ *for any* X*. It follows directly from symmetry that* $X \in \mathbb{Z}^\infty$ *if and only if* $\sigma X \in \mathbb{Z}^\infty$*.*

The ς *operator performs a cyclic operation- all floretion i's are replaced with j's, all j's replaced with k's and all k's replaced with i's.*

$$\begin{aligned}\varsigma X &= ibase(X)\overleftarrow{j} + jbase(X)\overleftarrow{k} + kbase(X)\overleftarrow{i} + basej(X)\overrightarrow{j} + basej(X)\overrightarrow{k} + basek(X)\overrightarrow{i} \\ &+ ibasei(X)\overleftrightarrow{jj} + jbasej(X)\overleftrightarrow{kk} + kbasek(X)\overleftrightarrow{ii} + jbasei(X)\overleftrightarrow{jk} + kbasei(X)\overleftrightarrow{ji} + ibasej(X)\overleftrightarrow{kj} \\ &+ kbasej(X)\overleftrightarrow{ki} + ibasek(X)\overleftrightarrow{ij} + jbasek(X)\overleftrightarrow{ik} + tes(X)\overleftrightarrow{ee}\end{aligned}$$

3.2 2nd Order Sequences

$$E = 0.25(\overleftarrow{i} + \overrightarrow{i} + ii + jj + kk + ee)$$



We are getting very close to Fibonacci numbers: if X is a "pure quaternion" of the form $A\overleftarrow{i} + B\overleftarrow{j} + C\overleftarrow{k}$, at least three proofs exist demonstrating that tesseq$[E * X]$ is at most a second order linear recurrence relation (it is in fact always a 2nd order linear recurrence relation, regardless of whether X is a pure quaternion or not).

1. Matrix methods- show that the characteristic equation of $E * X$ is of degree 2 or less.

2. The "by hand" procedure, below.
3. Using the Floretion Symbolic Calculator

Proposition 3.3 *(Pure Quaternions and 2nd Order Recurrence Relations) If* $X = A\overleftarrow{i} + B\overleftarrow{j} + C\overleftarrow{k}$ *and* $E = .25(\overleftarrow{i} + \overrightarrow{i} + ii + jj + kk + jk + kj + ee)$*, then*

$$a(n) = -A \cdot a(n-1) - B \cdot C \cdot a(n-2) \tag{13}$$

where $a(n) = base(X^n)$ *and* $base(X^n)$ *is any base coefficient of* X^n *(including "tes").* ***Proof.***

Use the following "tips":
Define $G_b = .25(\overleftarrow{j} - \overrightarrow{j} - ik + ki)$, $G_c = .25(\overleftarrow{k} - \overrightarrow{k} - ij + ji)$,

$F_a = .25(\overleftarrow{i} - \overrightarrow{i} - jk + kj)$, $F_b = .25(\overleftarrow{j} - \overrightarrow{j} + ik - ik)$, $F_c = .25(\overleftarrow{k} - \overrightarrow{k} - ij + ji)$

Moreover, let $G = .25(\overleftarrow{i} + \overrightarrow{i} + jk + kj)$ *and* $F = .25(ii + jj + kk + ee)$ *be as in the power sequence proposition and* $E = .25(\overleftarrow{i} + \overrightarrow{i} + ii + jj + kk + jk + kj + ee)$. *If* $X = x_0 \overleftarrow{i} + x_1 \overleftarrow{j} + x_2 \overleftarrow{k}$ *then the following holds:*

$$FxF_a = -x_0 F \tag{14}$$
$$FxF_b = -x_1 F \tag{15}$$
$$FxF_c = -x_2 F \tag{16}$$
$$GxG = -x_0 G \tag{17}$$
$$GxG_b = -x_1 G \tag{18}$$
$$GxG_c = -x_2 G \tag{19}$$
$$G_b x G_c = -FxE \tag{20}$$
$$Fx(Ex)^2 = Fx(-x_0 Ex - x_1 x_2 ee) \tag{21}$$

Proposition 3.4 *The General Case*
For any X *such that* $tes(X) = 0$, *i.e.* $X = A\overleftarrow{i} + B\overleftarrow{j} + C\overleftarrow{k} + D\overrightarrow{i} + E\overrightarrow{j} + F\overrightarrow{k} + G\overleftrightarrow{ii} + H\overleftrightarrow{jj} + I\overleftrightarrow{kk} + J\overleftrightarrow{ij} + K\overleftrightarrow{ik} + L\overleftrightarrow{ji} + M\overleftrightarrow{jk} + N\overleftrightarrow{ki} + O\overleftrightarrow{kj}$, *then*

$$a(n) = (-A-D+G+H+I+M+O)\cdot a(n-1) + ((A+D-M-O)(G+H+I) + (N+E-B-K)(J+C-F-L))\cdot a(n-2) \tag{22}$$

where $a(n) = base(X^n)$ *and* $base(X^n)$ *is any base coefficient of* X^n *(including "tes").* ***Proof****. Use the Floretion Symbolic Multiplier*

Explanation of Color-Coded Multiplication on Floret's Cube: The segments IE-IK, IK-JK, JK-JE, etc. are all light blue. This is because the result of multiplying the two ends of a segment are all of the form IE*IK = -EK, IK*JK = -KE, JK*JE = -EK, etc. "EK" or "KE" means one part white and one part blue. "EI" or "IE" means one part white and one part red. "EJ" or "JE" means one part white and one part green. "KJ" or "JK" means one part blue and one part green, etc..

There is an arrow on the line segment JK-IK pointing from JK to IK. This means multiplying JK*IK in that order will not produce an element with a negative sign. Indeed, JK*IK = EK

If there is no arrow on a segment that means the two ends commute.

If there is negative sign on a segment that means the two ends commute and multiplying them produces an element with a negative sign. Ex. EI*KI = KI*EI = -KE

The two segments are otherwise the same but one has a "sigma" that means the "swap operator" must be applied to one of them to make them the same. Ex. IJ*II = EK and JK*IK = KE look the same (apart from the sigma sign) and sigma(KE) = EK.

3.3 Force Transformations (incomplete)

The idea behind a force transformation is quite simple. What we want do to is take any sequence $(c(n))$ and transform it relative to a given floretion X. One of the easiest ways is this: Define $Y(0) = X + (c(0) - ves(X)) * ee$ and $Y(n+1) = Y(n) + (c(n) - ves(Y(n))) * ee$. Then we define

```
ves_transform_seq(X, c) = (ves(Y(0)), ves(Y(1)), ves(Y(2)), ves(Y(3)), ...)
jes_transform_seq(X, c) = (jes(Y(0)), jes(Y(1)), jes(Y(2)), jes(Y(3)), ...)
les_transform_seq(X, c) = (les(Y(0)), les(Y(1)), les(Y(2)), les(Y(3)), ...)
tes_transform_seq(X, c) = (tes(Y(0)), tes(Y(1)), tes(Y(2)), tes(Y(3)), ...)
etc.
```

What happens now? The reader should check to see that in this case

```
ves_transformed_seq(X, c)
```

is in fact our original sequence $(c(n))$! Hint: "ves" is a linear operator. Looking at the transformed jes, les, and tes sequences, we find that our former static identities are still valid. In particular, we have

```
ves_transform_seq(X, c) = jes_transfor_seq(X, c)
+ les_transform_seq(X, c) + tes_transform_seq(X, c)
```

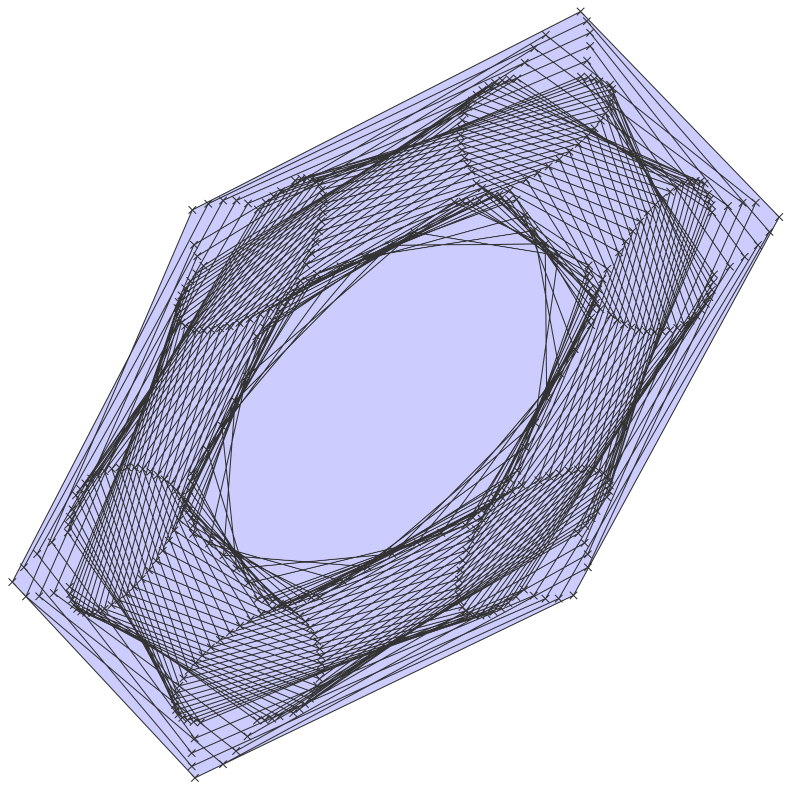
Often, the sequence being transformed will be the zero-sequence itself. Here is an OEIS example sequence A108300. Another example of the sequence $(1, -1, 1, -1, ...)$ being transformed is this A102129.

One of the neatest things about this is what happens when we work iteratively.

3.4 Necklaces (incomplete)

4 Structure in Fractional Parts

This section is currently written in an informal style.



Concerning the original question “Are these really 6 ellipses?” on the top graphic, it is still a bit unclear. Here is an excerpt from an post of mine last year:

```
Since 5 points define a conic section, I plotted the curve "Six ellipses",
switched to edit mode, and chose 5 points at random from one
of the six "ellipses".

This led to the equation:
-67.539x^2 + 84.408xy - 67.550y^2 - 521.317x - 817.342 + 325.815y = 0

 The other points also seem to fulfill this equation. For ex, let’s
 choose another point on the same "ellipse", say (-4.325, 1.340).
 Plugging this in to the above equation gives 0.10 = 0. That seems
 very good considering there’s some slight rounding going on.
 And speaking of rounding, changing the last point to (-4.325, 1.350)
 returns 2.1 = 0... a much bigger error. What is still unclear is what’s
```

```
happening as more and more points on the curve are plotted-
do these points approach a "real ellipse"?
```

4.1 Algorithms

Nearly all algoritms have one thing in common: given a floretion in the form $X = A\overleftarrow{i} + B\overleftarrow{j} + C\overleftarrow{k} + D\overrightarrow{i} + E\overrightarrow{j} + F\overrightarrow{k} + G\overleftrightarrow{ii} + H\overleftrightarrow{jj} + I\overleftrightarrow{kk} + J\overleftrightarrow{ij} + K\overleftrightarrow{ik} + L\overleftrightarrow{ji} + M\overleftrightarrow{jk} + N\overleftrightarrow{ki} + O\overleftrightarrow{kj} + P\overleftrightarrow{ee}$ we make use of the sum $A - [A] + B - [B] + C - [C] + ... + P - [P]$ where $[A]$ denotes the integer part of the number A. This sum can be used as a type of "steering wheel"- for example, if it is doubled and inserted back into the algorithm, ellipses may change their direction, their eccentricies, or turn into something completely different.

Perhaps one of the best examples for an introduction is this *Gerald's Diamonds* sequence: A108618. Notice that the sequence was created only using quaternions. To see what happens when we plot A108618(n) vs. A108618(n+1), see A108618, shifted, where the sequence is currently being used as an OEIS example.

Next, let's look at a similar example involving floretions:

Define

$X = -.5\overleftarrow{i} + \overleftarrow{j} - .5\overleftarrow{k} - .5\overrightarrow{i} + \overrightarrow{j} - .5\overrightarrow{k} + \overleftrightarrow{ii} - \overleftrightarrow{kk} + .5\overleftrightarrow{ij} + .5\overleftrightarrow{ji} - .5\overleftrightarrow{jk} - .5\overleftrightarrow{kj} + \overleftrightarrow{ee}$

$Y = X$

and

$Yd = -.5\overleftarrow{i} + .5\overleftarrow{j} - .5\overrightarrow{i} + .5\overrightarrow{j} - \overleftrightarrow{kk} - .5\overleftrightarrow{ik} - .5\overleftrightarrow{jk} - .5\overleftrightarrow{ki} - .5\overleftrightarrow{kj}'$

Note: the values of X and Yd do not change during the following steps.

STEP 1:

Multiply X with Y and set Y equal to this result ($Y = X \cdot Y$).

```
Pseudo-JavaCode:

mf = new MultiplyFlorets(X, Y);
System.arraycopy(mf.MultiplyFlorets(), 0, Y, 0, mf.MultiplyFlorets().length);


Result, First iteration:

Y =
 - 'i + 2'j - 'k - i' + 2j' - k' + 2'ii' - 2'kk' + 'ij' + 'ji' - 'jk' - 'kj' + 'ee'
```

STEP 2:

Multiply Y with Yd, set result equal Y.

```
Result, First iteration:
Y =
 + .5'i - 1.5'j + 'k + .5i' - 1.5j' + k' - 2'ii' + 'kk' - 'ij' - .5'ik' - 'ji' + .5'jk'
      - .5'ki' + .5'kj'
```

STEP 3:

Add the fractional parts of each basis vector coefficient from Y, set result equal to "sum".

```
Pseudo-JavaCode:

sum = 0; for (int g = 0; g <= 15; g++) sum = sum + Y[g] - (int) Y[g];

Note: each Y[g] represents the (real number) coefficient of one of the 16 basis vectors spar


Result, First iteration: sum =  0.5 - .5  + 0.5 - .5   - 0.5  + 0.5 - 0.5 + .5 = 0
```

STEP 4:

Add the value of "sum" from STEP 3 to the coefficient of the unit vector (i.e. "e") of Y.

```
Pseudo-JavaCode:  Y[15] = Y[15] + sum;



Result, First iteration:
Y = - .5'j + .5'k - .5j' + .5k' - 'ii' - .5'ij' - .5'ik' - .5'ji' - .5'ki'
```

STEP 5:

Set a(n) equal to the coefficient of the unit vector of Y, return to STEP 1 for next iteration.

Result, First iteration: a(1) = 0

The lines below represent the results after iterating the above steps. a(n) is given by the coefficient of the unit basis vector "e" at the end of each iteration. Interestingly, if we change STEP 4 to

"STEP 4":

Add *half* the value of "sum" from STEP 3 to the coefficient of the unit vector (i.e. "e") of Y.

Pseudo-JavaCode: Y[15] = Y[15] + sum/2;

we arrive at an integer sequence which mysteriously alternates (39, -39, 39, -39) for ever after 821 terms: A117154 OEIS

```
**************************************

-0.5'i 1.0'j -0.5'k -0.5i' 1.0j' -0.5k' 1.0'ii' 0.0'jj' -1.0'kk' 0.5'ij' 0.0'ik' 0.5'ji' -0.5'jk' 0.0'ki' -0.5'kj' 1.0e

a(0) = 1  (0-th iteration)

0.5'i -1.5'j 1.0'k 0.5i' -1.5j' 1.0k' -2.0'ii' 0.0'jj' 1.0'kk' -1.0'ij' -0.5'ik' -1.0'ji' 0.5'jk' -0.5'ki' 0.5'kj' 0.0e

a(1) = 0 (1st iteration)

-1.5'i 3.0'j -1.5'k -1.5i' 3.0j' -1.5k' 3.0'ii' 0.0'jj' -3.0'kk' 1.5'ij' 0.0'ik' 1.5'ji' -1.5'jk' 0.0'ki' -1.5'kj' -1.0e

a(2) = -1 (2nd iteration)
```

A126626

```
1.5'i -2.5'j 1.0'k 1.5i' -2.5j' 1.0k' -2.0'ii' 0.0'jj' 3.0'kk' -1.0'ij' 0.5'ik' -1.0'ji' 1.5'jk' 0.5'ki' 1.5'kj' 2.0e

-0.5'i 0.0'j 0.5'k -0.5i' 0.0j' 0.5k' -1.0'ii' 0.0'jj' -1.0'kk' -0.5'ij' -1.0'ik' -0.5'ji' -0.5'jk' -1.0'ki' -0.5'kj' -3.0e

-1.5'i 4.5'j -3.0'k -1.5i' 4.5j' -3.0k' 6.0'ii' 0.0'jj' -3.0'kk' 3.0'ij' 1.5'ik' 3.0'ji' -1.5'jk' 1.5'ki' -1.5'kj' 2.0e

4.5'i -10.0'j 5.5'k 4.5i' -10.0j' 5.5k' -11.0'ii' 0.0'jj' 9.0'kk' -5.5'ij' -1.0'ik' -5.5'ji' 4.5'jk' -1.0'ki' 4.5'kj' -1.0e

-6.5'i 13.5'j -7.0'k -6.5i' 13.5j' -7.0k' 14.0'ii' 0.0'jj' -13.0'kk' 7.0'ij' 0.5'ik' 7.0'ji' -6.5'jk' 0.5'ki' -6.5'kj' 2.0e

7.5'i -16.0'j 8.5'k 7.5i' -16.0j' 8.5k' -17.0'ii' 0.0'jj' 15.0'kk' -8.5'ij' -1.0'ik' -8.5'ji' 7.5'jk' -1.0'ki' 7.5'kj' 1.0e

-9.5'i 18.5'j -9.0'k -9.5i' 18.5j' -9.0k' 18.0'ii' 0.0'jj' -19.0'kk' 9.0'ij' -0.5'ik' 9.0'ji' -9.5'jk' -0.5'ki' -9.5'kj' 0.0e

8.5'i -17.0'j 8.5'k 8.5i' -17.0j' 8.5k' -17.0'ii' 0.0'jj' 17.0'kk' -8.5'ij' 0.0'ik' -8.5'ji' 8.5'jk' 0.0'ki' 8.5'kj' 3.0e

-8.5'i 15.5'j -7.0'k -8.5i' 15.5j' -7.0k' 14.0'ii' 0.0'jj' -17.0'kk' 7.0'ij' -1.5'ik' 7.0'ji' -8.5'jk' -1.5'ki' -8.5'kj' -2.0e

5.5'i -10.0'j 4.5'k 5.5i' -10.0j' 4.5k' -9.0'ii' 0.0'jj' 11.0'kk' -4.5'ij' 1.0'ik' -4.5'ji' 5.5'jk' 1.0'ki' 5.5'kj' 5.0e

-3.5'i 4.5'j -1.0'k -3.5i' 4.5j' -1.0k' 2.0'ii' 0.0'jj' -7.0'kk' 1.0'ij' -2.5'ik' 1.0'ji' -3.5'jk' -2.5'ki' -3.5'kj' -4.0e

-1.5'i 5.0'j -3.5'k -1.5i' 5.0j' -3.5k' 7.0'ii' 0.0'jj' -3.0'kk' 3.5'ij' 2.0'ik' 3.5'ji' -1.5'jk' 2.0'ki' -1.5'kj' 3.0e

5.5'i -12.5'j 7.0'k 5.5i' -12.5j' 7.0k' -14.0'ii' 0.0'jj' 11.0'kk' -7.0'ij' -1.5'ik' -7.0'ji' 5.5'jk' -1.5'ki' 5.5'kj' -4.0e

-8.5'i 19.0'j -10.5'k -8.5i' 19.0j' -10.5k' 21.0'ii' 0.0'jj' -17.0'kk' 10.5'ij' 2.0'ik' 10.5'ji' -8.5'jk' 2.0'ki' -8.5'kj' 1.0e

12.5'i -25.5'j 13.0'k 12.5i' -25.5j' 13.0k' -26.0'ii' 0.0'jj' 25.0'kk' -13.0'ij' -0.5'ik' -13.0'ji' 12.5'jk' -0.5'ki' 12.5'kj' -4.0e

-13.5'i 29.0'j -15.5'k -13.5i' 29.0j' -15.5k' 31.0'ii' 0.0'jj' -27.0'kk' 15.5'ij' 2.0'ik' 15.5'ji' -13.5'jk' 2.0'ki' -13.5'kj' -1.0e

17.5'i -34.5'j 17.0'k 17.5i' -34.5j' 17.0k' -34.0'ii' 0.0'jj' 35.0'kk' -17.0'ij' 0.5'ik' -17.0'ji' 17.5'jk' 0.5'ki' 17.5'kj' -2.0e

-16.5'i 34.0'j -17.5'k -16.5i' 34.0j' -17.5k' 35.0'ii' 0.0'jj' -33.0'kk' 17.5'ij' 1.0'ik' 17.5'ji' -16.5'jk' 1.0'ki' -16.5'kj' -3.0e

18.5'i -35.5'j 17.0'k 18.5i' -35.5j' 17.0k' -34.0'ii' 0.0'jj' 37.0'kk' -17.0'ij' 1.5'ik' -17.0'ji' 18.5'jk' 1.5'ki' 18.5'kj' 0.0e

-15.5'i 31.0'j -15.5'k -15.5i' 31.0j' -15.5k' 31.0'ii' 0.0'jj' -31.0'kk' 15.5'ij' 0.0'ik' 15.5'ji' -15.5'jk' 0.0'ki' -15.5'kj' -5.0e

15.5'i -28.5'j 13.0'k 15.5i' -28.5j' 13.0k' -26.0'ii' 0.0'jj' 31.0'kk' -13.0'ij' 2.5'ik' -13.0'ji' 15.5'jk' 2.5'ki' 15.5'kj' 2.0e

-10.5'i 20.0'j -9.5'k -10.5i' 20.0j' -9.5k' 19.0'ii' 0.0'jj' -21.0'kk' 9.5'ij' -1.0'ik' 9.5'ji' -10.5'jk' -1.0'ki' -10.5'kj' -7.0e

8.5'i -13.5'j 5.0'k 8.5i' -13.5j' 5.0k' -10.0'ii' 0.0'jj' 17.0'kk' -5.0'ij' 3.5'ik' -5.0'ji' 8.5'jk' 3.5'ki' 8.5'kj' 4.0e

-1.5'i 1.0'j 0.5'k -1.5i' 1.0j' 0.5k' -1.0'ii' 0.0'jj' -3.0'kk' -0.5'ij' -2.0'ik' -0.5'ji' -1.5'jk' -2.0'ki' -1.5'kj' -9.0e

-2.5'i 9.5'j -7.0'k -2.5i' 9.5j' -7.0k' 14.0'ii' 0.0'jj' -5.0'kk' 7.0'ij' 4.5'ik' 7.0'ji' -2.5'jk' 4.5'ki' -2.5'kj' 4.0e

11.5'i -25.0'j 13.5'k 11.5i' -25.0j' 13.5k' -27.0'ii' 0.0'jj' 23.0'kk' -13.5'ij' -2.0'ik' -13.5'ji' 11.5'jk' -2.0'ki' 11.5'kj' -7.0e

-15.5'i 34.5'j -19.0'k -15.5i' 34.5j' -19.0k' 38.0'ii' 0.0'jj' -31.0'kk' 19.0'ij' 3.5'ik' 19.0'ji' -15.5'jk' 3.5'ki' -15.5'kj' 4.0e
```

22.5’i -47.0’j 24.5’k 22.5i’ -47.0j’ 24.5k’ -49.0’ii’ 0.0’jj’ 45.0’kk’ -24.5’ij’ -2.0’ik’ -24.5’ji’ 22.5’jk’ -2.0’ki’ 22.5’kj’ -5.0e

****
****

A117154

A117154 OEIS (see above remark - This sequences is calculated similary, however, STEP 4 has been slightly changed)

-0.5’i 1.0’j -0.5’k -0.5i’ 1.0j’ -0.5k’ 1.0’ii’ 0.0’jj’ -1.0’kk’ 0.5’ij’ 0.0’ik’ 0.5’ji’ -0.5’jk’ 0.0’ki’ -0.5’kj’ 1.0e

0.5’i -1.5’j 1.0’k 0.5i’ -1.5j’ 1.0k’ -2.0’ii’ 0.0’jj’ 1.0’kk’ -1.0’ij’ -0.5’ik’ -1.0’ji’ 0.5’jk’ -0.5’ki’ 0.5’kj’ 0.0e

-1.5’i 3.0’j -1.5’k -1.5i’ 3.0j’ -1.5k’ 3.0’ii’ 0.0’jj’ -3.0’kk’ 1.5’ij’ 0.0’ik’ 1.5’ji’ -1.5’jk’ 0.0’ki’ -1.5’kj’ 0.0e

1.5’i -3.0’j 1.5’k 1.5i’ -3.0j’ 1.5k’ -3.0’ii’ 0.0’jj’ 3.0’kk’ -1.5’ij’ 0.0’ik’ -1.5’ji’ 1.5’jk’ 0.0’ki’ 1.5’kj’ 1.0e

-1.5’i 2.5’j -1.0’k -1.5i’ 2.5j’ -1.0k’ 2.0’ii’ 0.0’jj’ -3.0’kk’ 1.0’ij’ -0.5’ik’ 1.0’ji’ -1.5’jk’ -0.5’ki’ -1.5’kj’ -1.0e

0.5’i -2.0’j 1.5’k 0.5i’ -2.0j’ 1.5k’ -3.0’ii’ 0.0’jj’ 1.0’kk’ -1.5’ij’ -1.0’ik’ -1.5’ji’ 0.5’jk’ -1.0’ki’ 0.5’kj’ 0.0e

-2.5’i 5.0’j -2.5’k -2.5i’ 5.0j’ -2.5k’ 5.0’ii’ 0.0’jj’ -5.0’kk’ 2.5’ij’ 0.0’ik’ 2.5’ji’ -2.5’jk’ 0.0’ki’ -2.5’kj’ 1.0e

2.5’i -5.5’j 3.0’k 2.5i’ -5.5j’ 3.0k’ -6.0’ii’ 0.0’jj’ 5.0’kk’ -3.0’ij’ -0.5’ik’ -3.0’ji’ 2.5’jk’ -0.5’ki’ 2.5’kj’ 0.0e

-3.5’i 7.0’j -3.5’k -3.5i’ 7.0j’ -3.5k’ 7.0’ii’ 0.0’jj’ -7.0’kk’ 3.5’ij’ 0.0’ik’ 3.5’ji’ -3.5’jk’ 0.0’ki’ -3.5’kj’ 0.0e

3.5’i -7.0’j 3.5’k 3.5i’ -7.0j’ 3.5k’ -7.0’ii’ 0.0’jj’ 7.0’kk’ -3.5’ij’ 0.0’ik’ -3.5’ji’ 3.5’jk’ 0.0’ki’ 3.5’kj’ 1.0e

-3.5’i 6.5’j -3.0’k -3.5i’ 6.5j’ -3.0k’ 6.0’ii’ 0.0’jj’ -7.0’kk’ 3.0’ij’ -0.5’ik’ 3.0’ji’ -3.5’jk’ -0.5’ki’ -3.5’kj’ -1.0e

2.5’i -4.5’j 2.0’k 2.5i’ -4.5j’ 2.0k’ -4.0’ii’ 0.0’jj’ 5.0’kk’ -2.0’ij’ 0.5’ik’ -2.0’ji’ 2.5’jk’ 0.5’ki’ 2.5’kj’ 2.0e

-1.5’i 2.0’j -0.5’k -1.5i’ 2.0j’ -0.5k’ 1.0’ii’ 0.0’jj’ -3.0’kk’ 0.5’ij’ -1.0’ik’ 0.5’ji’ -1.5’jk’ -1.0’ki’ -1.5’kj’ -2.0e

-0.5’i 2.0’j -1.5’k -0.5i’ 2.0j’ -1.5k’ 3.0’ii’ 0.0’jj’ -1.0’kk’ 1.5’ij’ 1.0’ik’ 1.5’ji’ -0.5’jk’ 1.0’ki’ -0.5’kj’ 1.0e

2.5’i -5.5’j 3.0’k 2.5i’ -5.5j’ 3.0k’ -6.0’ii’ 0.0’jj’ 5.0’kk’ -3.0’ij’ -0.5’ik’ -3.0’ji’ 2.5’jk’ -0.5’ki’ 2.5’kj’ -2.0e

-3.5’i 8.0’j -4.5’k -3.5i’ 8.0j’ -4.5k’ 9.0’ii’ 0.0’jj’ -7.0’kk’ 4.5’ij’ 1.0’ik’ 4.5’ji’ -3.5’jk’ 1.0’ki’ -3.5’kj’ 0.0e

5.5’i -11.0’j 5.5’k 5.5i’ -11.0j’ 5.5k’ -11.0’ii’ 0.0’jj’ 11.0’kk’ -5.5’ij’ 0.0’ik’ -5.5’ji’ 5.5’jk’ 0.0’ki’ 5.5’kj’ -1.0e

-5.5’i 11.5’j -6.0’k -5.5i’ 11.5j’ -6.0k’ 12.0’ii’ 0.0’jj’ -11.0’kk’ 6.0’ij’ 0.5’ik’ 6.0’ji’ -5.5’jk’ 0.5’ki’ -5.5’kj’ 0.0e

6.5’i -13.0’j 6.5’k 6.5i’ -13.0j’ 6.5k’ -13.0’ii’ 0.0’jj’ 13.0’kk’ -6.5’ij’ 0.0’ik’ -6.5’ji’ 6.5’jk’ 0.0’ki’ 6.5’kj’ 0.0e

5 Additional Documentation When Using the Floretion Symbolic Multiplier

Below **EpowI** and **fib** are predefined (Python) floretion dictionaries. **T** is defined as the product of **EpowI** and **fib**.

```
>>> T = Mult(EpowI, fib)
ee   +0.25
ie   -0.25
je   -0.5
ke
ei   +0.25
ej
ek
ii   -0.25
jj   -0.25
kk   +0.25
ij   +0.5
ik   +0.5
ji   -0.5
jk   -0.25
ki
kj   +0.25

>>> seqFinder(T)

The first 10 terms of each sequence are as follows:
2jesseq: [-1, -2, -3, -5, -8, -13, -21, -34, -55, -89]
This sequence has no dynamic qualities.

4lesseq: [1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
4lesposseq: [5, 7, 12, 19, 31, 50, 81, 131, 212, 343]
2lesnegseq: [-2, -3, -5, -8, -13, -21, -34, -55, -89, -144]
Note the dynamic identity lespos + lesneg = les

4tesseq: [1, 3, 4, 7, 11, 18, 29, 47, 76, 123]
This sequence has no dynamic qualities.

1vesseq: [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
2vesposseq: [3, 5, 8, 13, 21, 34, 55, 89, 144, 233]
2vesnegseq: [-3, -5, -8, -13, -21, -34, -55, -89, -144, -233]
Note the dynamic identity vespos + vesneg = ves
```

The above implies that

```
0.25*[1, 1, 2, 3, 5, 8, 13, 21, 34, 55] =
0.25*[5, 7, 12, 19, 31, 50, 81, 131, 212, 343] +
0.5*[-2, -3, -5, -8, -13, -21, -34, -55, -89, -144]
```

In the case of ves, it is not hard to imagine an egg being cracked open and/or getting something for nothing when two dynamic identities add to the zero sequence!

To see which basis vectors are associated with jesleft, for example, use the command

```
>>> returnsetofvectors("jesleft")
['ke', 'je', 'ie']
>>> returnsetofvectors("ibase")
['ie']
>>> returnsetofvectors("basei")
['ei']
```

This means that jesleft sums up over the coefficients of the basis vectors ['ke', 'je', 'ie'], namely: A, B, C. Now, a quick glance confirms that

```
jesleft(X) + jesright(X) = jes(X)
```

The command `testequality(listofstrings1, listofstrings2)` can be used here:

```
>>> testequality(["jesright","jesleft"], ["jes"])
True
>>> testequality(["tes","les", "jes"], ["ves"])
True
```

The function is particularly handy when creating your own identities or when checking lengthier ones. If you wish to create your own identity- say

```
myown = ibase + basej
```

it is merely necessary to look for the dictionary labled "staticidentities" in the original code and add the string "myown" to the list next to the vectors `'ie'` and `'ej'`. Some useful pre-defined identities are given below.

```
>>> testequality(["emI", "emJ", "emK"],
["fam", "fam", "fam", "tes", "tes", "tes", "jes", "mix"])
True
```

This refers to the identity : emI + emJ + emK = 3fam + 3tes + jes + mix

```
>>> testequality(["les"], ["fam", "mix"])
True

>>> testequality(["emI"], ["ibase", "basei", "ibasei", "jbasej",
"kbasek", "jbasek", "kbasej", "tes"])
True

>>> testequality(["diaI", "diaJ", "diaK"], ["jes", "fam"])
True
```

More examples...

```
>>> seqFinder(T)

(jes + les + tes = ves)
>>> T = Mult(ELucI,fib)
2*Fib(1,1) + Fib(1,-1) = Luc(1,3)

>>> T2 = Mult(ELucI,fib2)
Fib(1,1) + Luc(1,3) = 2*Fib(1,2)

>>> T3 = Mult(EFibI,fib3)
Luc(1,3) = 2*Fib(0,1) + Fib(1,1)

(vespos = jespos + lespos + tespos)
>>> T = Mult(ELucI,fib)
2*Luc(4,7) = Luc(3,4) + 5*Fib(1,2)


>>> T2 = Mult(ELucI,fib2)
2*Fib(3,5) = Luc(1,3) + O2(5,7)


>>> T3 = Mult(EFibI,fib3)
2*Fib(2,3) + O2(5,7) + Luc(1,3) = 2*Fib(5,8)


(vesneg = jesneg + lesneg + tesneg)
>>> T = Mult(ELucI,fib)
2*Luc(4,7) = 2*Fib(3,5) + 2*Fib(1,2)
-> Luc(4,7) = Fib(3,5) + Fib(1,2)

>>> T2 = Mult(ELucI,fib2)
2*Fib(1,2) + 2*Fib(2,3) = 2*Fib(3,5)
-> Fib(1,2) + Fib(2,3) = Fib(3,5) (not new)
```

```
 >>> T = Mult(EpowI,fib)
4*Fib(0,1) + 4*Fib(0,1) = 3*Fib(-1,1) + 3*Luc(1,3) - 2*Fib(1,1) - 2*Fib(-1,1)
-> 8*Fib(0,1) = Fib(-1,1) + 3*Luc(1,3) - 2*Fib(1,1)

>>> T3 = Mult(EFibI,fib3)
4*Fib(1,1) - 4*Fib(1,0) = 3*Luc(1,3) - 3*Luc(1,3) + 4*Fib(0,1) (cancels, nothing new)


jesright + jesleft = jes
>>> T2 = Mult(ELucI,fib2)
3*Fib(0,1) + Luc(2,1) = 2*Fib(1,2)


 emJ = emJpos + emJneg
 >>> T3 = Mult(EFibI,fib3)
 2*Fib(1,0) = 02(4,5) - 02(2,5)

The first 10 terms of each sequence are as follows:
2jesseq: [0, 1, 1, 2, 3, 5, 8, 13, 21, 34]
2jesposseq: [2, 3, 5, 8, 13, 21, 34, 55, 89, 144]
1jesnegseq: [-1, -1, -2, -3, -5, -8, -13, -21, -34, -55]
Note the dynamic identity jespos + jesneg = jes

4lesseq: [-1, -1, -2, -3, -5, -8, -13, -21, -34, -55]
4lesposseq: [5, 7, 12, 19, 31, 50, 81, 131, 212, 343]
2lesnegseq: [-3, -4, -7, -11, -18, -29, -47, -76, -123, -199]
Note the dynamic identity lespos + lesneg = les

4tesseq: [1, 3, 4, 7, 11, 18, 29, 47, 76, 123]
This sequence has no dynamic qualities.

1vesseq: [0, 1, 1, 2, 3, 5, 8, 13, 21, 34]
2vesposseq: [5, 8, 13, 21, 34, 55, 89, 144, 233, 377]
2vesnegseq: [-5, -6, -11, -17, -28, -45, -73, -118, -191, -309]
Note the dynamic identity vespos + vesneg = ves

4jesrightseq: [-1, 0, -1, -1, -2, -3, -5, -8, -13, -21]
2jesrightposseq: [1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
4jesrightnegseq: [-3, -2, -5, -7, -12, -19, -31, -50, -81, -131]
Note the dynamic identity jesrightpos + jesrightneg = jesright

4jesleftseq: [1, 2, 3, 5, 8, 13, 21, 34, 55, 89]
2jesleftposseq: [1, 2, 3, 5, 8, 13, 21, 34, 55, 89]
4jesleftnegseq: [-1, -2, -3, -5, -8, -13, -21, -34, -55, -89]
Note the dynamic identity jesleftpos + jesleftneg = jesleft
```

```
2mixseq: [0, 1, 1, 2, 3, 5, 8, 13, 21, 34]
2mixposseq: [1, 2, 3, 5, 8, 13, 21, 34, 55, 89]
2mixnegseq: [-1, -1, -2, -3, -5, -8, -13, -21, -34, -55]
Note the dynamic identity mixpos + mixneg = mix

4famseq: [-1, -3, -4, -7, -11, -18, -29, -47, -76, -123]
4famposseq: [3, 3, 6, 9, 15, 24, 39, 63, 102, 165]
2famnegseq: [-2, -3, -5, -8, -13, -21, -34, -55, -89, -144]
Note the dynamic identity fampos + famneg = fam

1emIseq: [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
2emIposseq: [3, 4, 7, 11, 18, 29, 47, 76, 123, 199]
2emInegseq: [-3, -4, -7, -11, -18, -29, -47, -76, -123, -199]
Note the dynamic identity emIpos + emIneg = emI

1emJseq: [-1, 0, -1, -1, -2, -3, -5, -8, -13, -21]
2emJposseq: [2, 5, 7, 12, 19, 31, 50, 81, 131, 212]
2emJnegseq: [-4, -5, -9, -14, -23, -37, -60, -97, -157, -254]
Note the dynamic identity emJpos + emJneg = emJ

1emKseq: [1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
2emKposseq: [4, 5, 9, 14, 23, 37, 60, 97, 157, 254]
2emKnegseq: [-2, -3, -5, -8, -13, -21, -34, -55, -89, -144]
Note the dynamic identity emKpos + emKneg = emK

4diaIseq: [-5, -5, -10, -15, -25, -40, -65, -105, -170, -275]
This sequence has no dynamic qualities.

4diaJseq: [1, 3, 4, 7, 11, 18, 29, 47, 76, 123]
4diaJposseq: [3, 5, 8, 13, 21, 34, 55, 89, 144, 233]
2diaJnegseq: [-1, -1, -2, -3, -5, -8, -13, -21, -34, -55]
Note the dynamic identity diaJpos + diaJneg = diaJ

4diaKseq: [3, 1, 4, 5, 9, 14, 23, 37, 60, 97]
1diaKposseq: [1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
4diaKnegseq: [-1, -3, -4, -7, -11, -18, -29, -47, -76, -123]
Note the dynamic identity diaKpos + diaKneg = diaK

2diaItesseq: [-2, -1, -3, -4, -7, -11, -18, -29, -47, -76]
4diaItesposseq: [1, 3, 4, 7, 11, 18, 29, 47, 76, 123]
4diaItesnegseq: [-5, -5, -10, -15, -25, -40, -65, -105, -170, -275]
Note the dynamic identity diaItespos + diaItesneg = diaItes

2diaJtesseq: [1, 3, 4, 7, 11, 18, 29, 47, 76, 123]
1diaJtesposseq: [1, 2, 3, 5, 8, 13, 21, 34, 55, 89]
```

```
2diaJtesnegseq: [-1, -1, -2, -3, -5, -8, -13, -21, -34, -55]
Note the dynamic identity diaJtespos + diaJtesneg = diaJtes

1diaKtesseq: [1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
4diaKtesposseq: [5, 7, 12, 19, 31, 50, 81, 131, 212, 343]
4diaKtesnegseq: [-1, -3, -4, -7, -11, -18, -29, -47, -76, -123]
Note the dynamic identity diaKtespos + diaKtesneg = diaKtes
```

6 Acknowledgements and Closing Remarks

It can be said that the current paper is the result of "7 years of casual note taking in my free time". Fortunately, there has been encouragement from others concerning this work and many useful tips along the way. Special thanks to Prof. Irene Pieper-Seier for her initial proofreading in 2002 and pointing out that the found group of 32 elements was isomorph to the Quaternion Factor Space $\mathcal{F} = \mathcal{Q} \times \mathcal{Q}/\{(1,1),(-1,-1)\}$. Thanks also to Prof. Edwin Clark for providing a Maple worksheet to the author which clarified how the 16 (positive) base elements span the algebra of 4×4 matrices and that this algebra is the tensor product of H with H (where H is the algebra of quaternions). Note: not all basis vectors are shown, below:

$$IE = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} EI = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$JE = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} EJ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$KE = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} EK = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$EE = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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